An Axiomatization of Shapley Values of Games on Set Systems

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Abstract. An axiomatization of a generalized Shapley value of games is proposed. The authors follow Faigle and Kern, in the sense that our basic material is the maximal chains of the underlying set system. This generalized Shapley value may have applicability to the game on set systems which satisfy the condition of a sort of normality.

1 Introduction

Let $X = \{1, 2, \ldots, n\}$ be a set consisted of n players. Then a subset of X is called a coalition. A function which shows the profit by any coalition $v : 2^X \to \mathbb{R}$ is called a cooperative game. The solution of the game is a function from the whole set of the game to n dimension real vector which measures each player's contribution or share-out. The Shapley value and the Banzhaf value are known as the solution, and they are characterized by natural axiomatizations [3][4]. In this paper we shall show the axiomatization of the generalized Shapley value. Faigle and Kern have generalized the Shapley value using the concept of the maximal chain [5], it can be applied to the multi-choice game. Algaba et al. have also generalized it using the concept of the interior which can be applied to the game defined on antimatroid set systems, and they have given its axiomatizations [1]. Using these generalized Shapley value, we can obtain solutions of the bi-capacity and the multi-choice game and so on. We shall make Faigle and Kern's Shapley value more general and also show its axiomatization.

2 Set System and Shapley Value of a Game on It

We begin by introducing some notations and definitions. Throughout this paper, we consider a finite universal set $X = \{1, 2, \ldots, n\}$, $n \geq 1$, and 2^X denotes the power set of X. Let us consider $\mathfrak{S}$ a subset of 2^X which contains $\emptyset$ and X. Then we call $(X, \mathfrak{S})$ (or simply $\mathfrak{S}$ if no confusion occurs) a *set system*. A set system endowed with inclusion is a particular case of a *partially ordered set* $(\mathfrak{S}, \subseteq)$, i.e., a set $\mathfrak{S}$ endowed with a partial order (reflexive, antisymmetric and transitive) as $\subseteq$.

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Let $A, B \in \mathfrak{S}$. We say that A is *covered* by B, and write $A \prec B$ or $B \succ A$, if $A \subsetneq B$ and $A \subseteq C \subsetneq B$ together with $C \in \mathfrak{S}$ imply $C = A$.

Definition 1 (maximal chain of set system). *Let $\mathfrak{S}$ be a set system. We call $\mathscr{C}$ a* maximal chain of $\mathfrak{S}$ *if $\mathscr{C} = (C_0, C_1, \dots, C_m)$ satisfies $\emptyset = C_0 \prec C_1 \prec \cdots \prec C_m = X, C_i \in \mathfrak{S}, i = 0, \dots, m$.*

The length of the maximal chain $\mathscr{C} = (C_0, C_1, \dots, C_m)$ is m. We denote the set of all m-length maximal chains of $\mathfrak{S}$ by $\Gamma_m(\mathfrak{S})$, $1 \le m \le n$.

Example 1. Let $X := \{1,2,3\}, \mathfrak{S}_1 := (\{\emptyset, \{1\}, \{1,2\}, \{1,3\}, \{2,3\}, X\}$. Then the maximal chains of $\mathfrak{S}_1$ are $\mathscr{C}_1 := \{\emptyset, \{1\}, \{1,2\}, X\}, \mathscr{C}_2 := \{\emptyset, \{1\}, \{1,3\}, X\}$, $\mathscr{C}_3 := \{\emptyset, \{2,3\}, X\}, \Gamma_2(\mathfrak{S}) = \{\mathscr{C}_3\}$ and $\Gamma_3(\mathfrak{S}) = \{\mathscr{C}_1, \mathscr{C}_2\}$ (Fig. 1).

Remark 1. $(X, 2^X)$ has $n!$ maximal chains and all of their length are n.

Definition 2 (totally ordered set system). *We say that $(X, \mathfrak{S})$ is a* totally ordered set system *if for any $A, B \in \mathfrak{S}$, either $A \subseteq B$ or $A \supsetneq B$.*

If $(X, \mathfrak{S})$ is a totally ordered set system, then it has only one maximal chain which length is n.

Definition 3 (normal set system). *We say that $(X, \mathfrak{S})$ is a* normal set system *if for any $A \in \mathfrak{S}$ there exists $\mathscr{C} \in \Gamma_n(\mathfrak{S})$ satisfying $A \in \mathscr{C}$.*

Example 2. $(\{1,2,3\}, \{\emptyset, \{1\}, \{1,2\}, \{1,3\}, \{2,3\}, \{1,2,3\}\})$ is not a normal set system, because there are not any 3-length maximal chains which includes $\{2,3\}$ (Fig. 1).

Remark 2. Normality does not mean that all length of maximal chains are n. In fact, $(\{1,2,3,4\}, \{\emptyset, \{1\}, \{3\}, \{1,2\}, \{3,4\}, \{1,2,3\}, \{2,3,4\}, \{1,2,3,4\}\})$, which has 3-length maximal chain $(\emptyset, \{3\}, \{1,2,3\}, \{1,2,3,4\})$, is a normal set system (Fig. 2).

Definition 4 (game on a set system). *Let $(X, \mathfrak{S})$ be a set system. A function $v : \mathfrak{S} \to \mathbb{R}$ is a* game on $(X, \mathfrak{S})$ *if it satisfies $v(\emptyset) = 0$.*

Definition 5 (Shapley value of game on $(X, 2^X)$[3]). *Let v be a game on $(X, 2^X)$. The* Shapley value of v, *$\Phi(v) = (\phi^1(v), \dots, \phi^n(v)) \in [0,1]^n$ is defined by*

$$\phi^i(v) := \sum_{A \subseteq X \setminus \{i\}} \gamma^n_{|A|}(v(A \cup \{i\}) - v(A)), \; i = 1, \dots, n,$$

where

$$\gamma^n_k := \frac{(n-k-1)!\; k!}{n!}.$$

Remark that $\sum_{i=1}^n \phi^i(v) = v(X)$ holds. The Shapley value can be represented by using the maximal chains as follows.

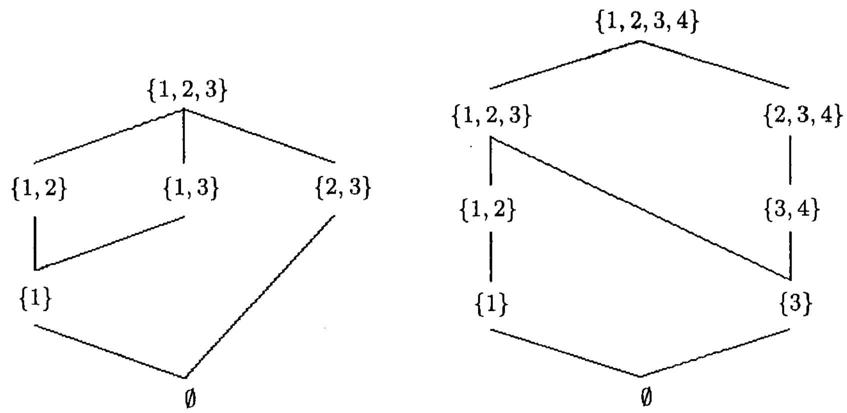

Fig. 1. set systems

Proposition 1. *Fix arbitrarily* $i \in X$. *For any* $\mathscr{C} \in \Gamma_n(2^X)$, *there exists a unique* $A_{\mathscr{C}} \in \mathscr{C}$ *such that* $i \notin A_{\mathscr{C}}$ *and* $A_{\mathscr{C}} \cup \{i\} \in \mathscr{C}$, *and*

$$\phi^i(v) = \frac{1}{n!} \sum_{\mathscr{C} \in \Gamma_n(2^X)} (v(A_{\mathscr{C}} \cup \{i\}) - v(A_{\mathscr{C}}))$$

holds.

The fact is well known in the game theory. We give a proof of Proposition 1 for the sake of completeness.

Proof. $|\Gamma_n(2^X)| = n!$ holds. First, we show that for any $\mathscr{C} \in \Gamma_n(2^X)$, there is an $A_{\mathscr{C}} \in \mathscr{C}$ such that $i \notin A_{\mathscr{C}}$ and $A_{\mathscr{C}} \cup \{i\} \in \mathscr{C}$. Fix $\mathscr{C} = (C_0, C_1, \ldots, C_m) \in \Gamma_n(2^X)$. We have for $k = 1, \ldots, m$, $|C_k \setminus C_{k-1}| = 1$ so that $m = n$ holds. We have $C_k \setminus C_{k-1} \neq C_j \setminus C_{j-1}$ for $k < j$ because if $C_k \setminus C_{k-1} = C_j \setminus C_{j-1}$ then $C_j \supseteq C_k \setminus C_{k-1}$. But since $k < j$, $C_{j-1} \supseteq C_k$ therefore $C_{j-1} \supseteq C_k \setminus C_{k-1}$ which is a contradiction. Hence for any $i \in X$, there is an $A_{\mathscr{C}} \in \mathscr{C}$, which satisfies $i \notin A_{\mathscr{C}}$ and $A_{\mathscr{C}} \cup \{i\} \in \mathscr{C}$.

Next we show that for $A \subseteq X \setminus \{i\}$, the number of chains which include $A \cup \{i\}$ and A is $(n - |A| - 1)!|A|!$. Fix arbitrarily $i \in X$. Number of chains from $A \cup \{i\}$ to X is $(n - |A| - 1)!$ and chains from $\emptyset$ to A is $|A|!$. Hence the number of chains which include $A \cup \{i\}$ and A is $|A|! \cdot (n - |A| - 1)!$. Therefore

$$\frac{1}{n!} \sum_{\mathscr{C} \in \Gamma_n(2^X)} (v(A_{\mathscr{C}} \cup \{i\}) - v(A_{\mathscr{C}}))$$
$$= \frac{(n - |A| - 1)! \cdot |A|!}{n!} \sum_{A \in X \setminus \{i\}} (v(A \cup \{i\}) - v(A)),$$

which completes the proof.

Faigle and Kern had generalized the Shapley value for applying the multi-choice game using the concept of the maximal chain. We extend their Shapley value for applying to more general cases.

Definition 6 (Shapley value of game on set system). *Let v be a game on a normal set system $(X, \mathfrak{S})$. The Shapley value of v, $\Phi(v) = (\phi^1(v), \ldots, \phi^n(v)) \in \mathbb{R}^n$ is defined by*

$$\textbf{(FK)} \qquad \phi^i_{\mathrm{FK}}(v) := \frac{1}{|\Gamma_n(\mathfrak{S})|} \sum_{\mathscr{C} \in \Gamma_n(\mathfrak{S})} (v(A_{\mathscr{C}} \cup \{i\}) - v(A_{\mathscr{C}})),$$

where $A_{\mathscr{C}} := A \in \mathscr{C} \in \Gamma_n(\mathfrak{S})$ such that $i \notin A$ and $A \cup \{i\} \in \mathscr{C}$.

We discuss the domains of Φ. Let $(X, \mathfrak{S})$ be a normal set system and let v be a game on $(X, \mathfrak{S})$. Then we call $(X, \mathfrak{S}, v)$ a game space. Let Σ_n be the set of all normal set systems of $X := \{1, 2, \ldots, n\}$ and let $\Delta_{\mathfrak{S}}$ be the set of all game spaces defined on a normal set system $(X, \mathfrak{S})$. The domain of Φ is $\Delta := \bigcup_{n=1}^{\infty} \bigcup_{\mathfrak{S} \in \Sigma_n} \Delta_{\mathfrak{S}}$, and Φ is a function defined on Δ to $\mathbb{R}^n$. We denote simply $\Phi(v)$ and $\phi^i(v)$ instead of $\Phi(X, \mathfrak{S}, v)$ and $\phi^i(v)$ as far as no confusion occurs.

We introduce further concepts about games, which will be useful for stating axioms.

Definition 7 (dual game). *Let v be a game on $(X, \mathfrak{S})$. Then the* dual game *of v is defined on $\mathfrak{S}^d := \{A \in 2^X \mid A^c \in \mathfrak{S}\}$ by $v^d(A) := 1 - v(A^c)$ for any $A \in \mathfrak{S}^d$, where $A^c := X \setminus A$.*

Definition 8 (permutation of v). *Let v be a game on $(X, \mathfrak{S})$ and π be a permutation on X. Then the* permutation *of v by π is defined on $\pi(\mathfrak{S}) := \{\pi(A) \in 2^X \mid A \in \mathfrak{S}\}$ by $\pi \circ v(A) := v(\pi^{-1}(A))$.*

Let us consider a chain of length 2 as a set system, denoted by $\mathbf{2}$ (e.g., $\{\emptyset, \{1\}, \{1,2\}\}$), and a game $v^{\mathbf{2}}$ on it. We denote by the triplet $(0, u, t)$ the values of $v^{\mathbf{2}}$ along the chain. We suppose $\mathbf{2} := \{\emptyset, \{1\}, \{1,2\}\}$ unless otherwise noted.

Definition 9 (embedding of $v^{\mathbf{2}}$). *Let v be a game on a totally ordered normal set system $(X, \mathfrak{S})$, where $\mathfrak{S} := \{C_0, \ldots, C_n\}$ such that $C_{i-1} \prec C_i$, $i = 1, \ldots, n$, and let $v^{\mathbf{2}} := (0, u, t), t \neq 0$, be a game on $\mathbf{2}$. Then for $C_k \in \mathfrak{S}$, v^{C_k} is called the* embedding *of $v^{\mathbf{2}}$ into v at C_k and defined on the totally ordered normal set system $(X^{C_k}, \mathfrak{S}^{C_k})$ by*

$$v^{C_k}(A) := \begin{cases} v(C_j), & \text{if } A = C_j, j < k, \\ v(C_{k-1}) + \frac{u}{t} \cdot \left(v(C_k) - v(C_{k-1})\right), & \text{if } A = C'_k, \\ v(C_{j-1}), & \text{if } A = C'_j, j > k, \end{cases} \tag{1}$$

where $\{i_k\} := C_k \setminus C_{k-1}, i'_k \neq i''_k$, $(X \setminus \{i_k\}) \cap \{i'_k, i''_k\} = \emptyset, X^{C_k} := (X \setminus \{i_k\}) \cup \{i'_k, i''_k\}, C'_k := (C_k \setminus \{i_k\}) \cup \{i'_k\}, C'_j := (C_{j-1} \setminus \{i_k\}) \cup \{i'_k, i''_k\}$ for $j > k$, and $\mathfrak{S}^{C_k} := \{C_0, \ldots, C_{k-1}, C'_k, C'_{k+1}, \ldots, C'_{n+1}\}$.

Remark that more properly, the dual game of v is the dual game space of the game space $(X, \mathfrak{S}, v)$ which is defined by $(X, \mathfrak{S}, v)^d := (X, \mathfrak{S}^d, v^d)$ with $\mathfrak{S}^d := \{A^c \in 2^X \mid A \in \mathfrak{S}\}$, the permutation of v is the permutation of the game space $(X, \mathfrak{S}, v)$ which is defined by $(X, \mathfrak{S}, v)^\pi := (X, \pi(\mathfrak{S}), \pi \circ v)$, and the embedding of $(0, u, t), t \neq 0$, into v is the embedding of the game space $(\{1, 2\}, \mathbf{2}, (0, u, t))$ into the game space $(X, \mathfrak{S}, v)$, and it is defined by $(X, \mathfrak{S}, v)^{C_k} := (X^{C_k}, \mathfrak{S}^{C_k}, v^{C_k})$.

3 Axiomatization of the Shapley Value of Games

We introduce six axioms for the Shapley value of games on normal set systems.

Axiom 1 (continuity of v^2). *The function $\phi^1(0, u, t)$ is continuous with respect to u.*

Axiom 2 (efficiency of v^2). *For any game $v^2 = (0, u, t)$ on **2**, $\phi^1(0, u, t) + \phi^2(0, u, t) = t = v(X)$ holds.*

Axiom 3 (dual invariance of v^2). *For any game $v^2 = (0, u, t)$ on **2**, $\Phi(0, u, t) = \Phi(0, u, t)^d$ holds.*

Axiom 4 (embedding efficiency). *Let $(X, \mathfrak{S})$ be a totally ordered normal set system and let $\mathfrak{S} := \{C_0, \ldots, C_n\}, C_{i-1} \prec C_i, i = 1, \ldots, n$. Then for any v on $(X, \mathfrak{S})$, any $(0, u, t), t \neq 0$, and any $C_k \in \mathfrak{S}$, $\phi^i(v^{C_k}) = \phi^i(v)$ for any $i \neq i'_k, i''_k$, $\phi^{i'_k}(v^{C_k}) = \phi^{i_k}(v) \cdot \phi^1(0, u, t)/t$ and $\phi^{i''_k}(v^{C_k}) = \phi^{i_k}(v) \cdot \phi^2(0, u, t)/t$ hold, where $\{i_k\} := C_k \setminus C_{k-1}$.*

Axiom 5 (convexity). *Let $(X, \mathfrak{S})$, $(X, \mathfrak{S}_1)$ and $(X, \mathfrak{S}_2)$ be normal set systems satisfying $\Gamma_n(\mathfrak{S}_1) \cup \Gamma_n(\mathfrak{S}_2) = \Gamma_n(\mathfrak{S})$ and $\Gamma_n(\mathfrak{S}_1) \cap \Gamma_n(\mathfrak{S}_2) = \emptyset$ and v be a game on $\mathfrak{S}$. Then there exists $\alpha \in]0, 1[$ satisfying that for every game v on $\mathfrak{S}$ and for every $i \in X$, $\phi^i(v) = \alpha\phi^i(v|_{\mathfrak{S}_1}) + (1 - \alpha)\phi^i(v|_{\mathfrak{S}_2})$.*

Axiom 6 (permutation invariance). *Let $(X, \mathfrak{S})$ be a normal set system and v be a game on $(X, \mathfrak{S})$. Then for any permutation π on X satisfying $\pi(\mathfrak{S}) = \mathfrak{S}$, $\phi^i(v) = \phi^{\pi(i)}(\pi \circ v), i = 1, \ldots, n$ holds.*
Then we obtain the following theorem.

Theorem 1. *Let v be a game on a normal set system $(X, \mathfrak{S})$. Then there exists a unique function satisfying* Axioms 1, 2, 3, 4, 5 *and* 6, *and it is given by* (FK).

Now, we discuss in detail the above axioms.

3.1 Efficiency of v^2

More generally, for any game on a normal set system, Axiom 2 holds.

Proposition 2. *Let $(X, \mathfrak{S})$ be a normal set system. Then for any game on $(X, \mathfrak{S})$, $\sum_{i=1}^n \phi^i_{\mathrm{FK}}(v) = v(X)$ holds.*

3.2 Dual Invariance

More generally, for any game on a normal set system, $\Phi(v)$ is dual invariant.

Proposition 3. *Let $(X, \mathfrak{S})$ be a normal set system. Then for any v on $(X, \mathfrak{S})$, $\Phi_{\mathrm{FK}}(v^d) = \Phi_{\mathrm{FK}}(v)$.*

3.3 Embedding Efficiency

Let v be a game on a totally ordered normal set system $\mathfrak{S} := \{C_0, \ldots, C_n\}$ such that $C_{i-1} \prec C_i$, $i = 1, \ldots, n$, Then the embedding at C_k into v by $(0, u, t)$ means that $i_k := C_k \backslash C_{k-1}$ is splitted to $\{i'_k, i''_k\}$. Axiom 4 implies $\phi^{i'_k}(v^{C_k}) + \phi^{i''_k}(v^{C_k}) = \phi^{i_k}(v)$ and $\phi^i(v^{C_k}) = \phi^i(v)$ for $i \neq i', i''$, so that Axiom 4 is natural property in the meaning of the contributions of i'_k and i''_k.

4 Application to Game on Lattice

The *lattice* $(L, \leq)$ is a partially ordered set such that for any pair $x, y \in L$, there exist a least upper bound $x \vee y$ (supremum) and a greatest lower bound $x \wedge y$ (infimum) in L. Consequently, for finite lattices, there always exist a greatest element (supremum of all elements) and a least element (infimum of all elements), denoted by $\top, \bot$ (see [2]). Our approach may have applicability to games defined on lattices which satisfy a kind of normality by the translation from lattices to set systems (cf. [6]).

Definition 10 (game on lattice). *Let $(L, \leq)$ be a finite lattice with least element denoted by $\bot$. A* game *on L is a function $v : L \longrightarrow I\!R$ satisfying $v(\bot) = 0$.*

Evidently the set system is not necessarily a lattice. Moreover, the normal set system is not necessarily a lattice. Indeed, take $X := \{1, 2, 3, 4\}$ and $\mathfrak{S} := \{\emptyset, \{1\}, \{4\}, \{1, 2\}, \{1.3\}, \{2, 4\}, \{3, 4\}, \{1, 2, 4\}, \{1, 3, 4\}, X\}$. Then, $(X, \mathfrak{S})$ is a normal set system, but it is not a lattice, because there is not the supremum of $\{1\}$ and $\{4\}$ (Fig. 2).

Definition 11 (join-irreducible element). *An element $x \in (L, \leq)$ is* join-irreducible *if for all $a, b \in L$, $x \neq \bot$ and $x = a \vee b$ implies $x = a$ or $x = b$.*

We denote the set of all join-irreducible elements of L by $\mathcal{J}(L)$.

The mapping η for any $a \in L$, defined by

$$\eta(a) := \{x \in \mathcal{J}(L) \mid x \leq a\}$$

is a lattice-isomorphism of L onto $\eta(L) := \{\eta(a) \mid a \in L\}$, that is, $(L, \leq) \cong (\eta(L), \subseteq)$. (Fig. 3)

Translating lattices which is underlying space of a game v to set systems, we obtain a set of players as $\mathcal{J}(L)$ and a set system as $\eta(L)$ and we can apply Definition 6 to a game on them if that is the case where the set system is normal.

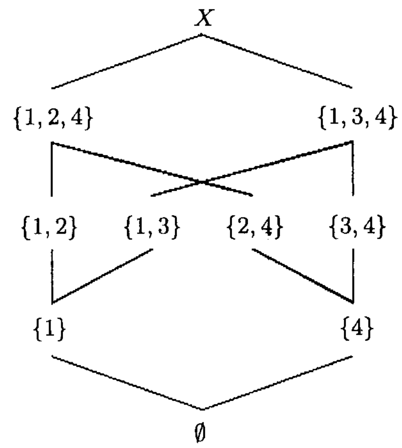

Fig. 2.

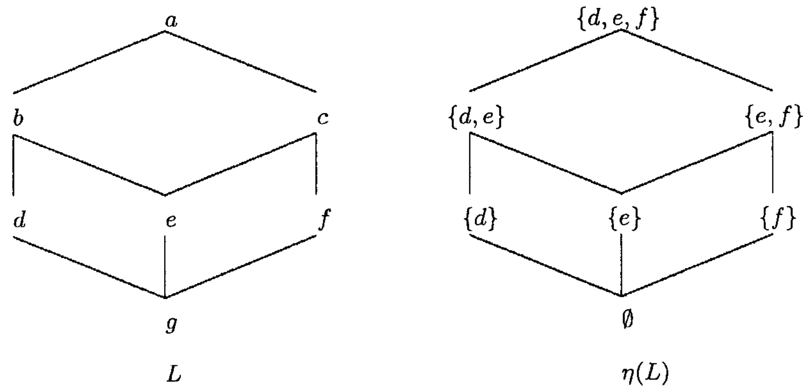

Fig. 3. Translation of lattice

In this paper we treat games defined on normal set systems. If the underlying space is not normal, Definition 6 can not be applied to such games and this fact is natural. Because $\phi^i(v)$ means a sort of the contribution of player i and is calculated as an average of i's contributions $v(A \cup \{i\}) - v(A)$. For instance, let $X := \{1, 2, 3\}$ and $\mathfrak{S}_1 := \{\emptyset, \{1\}, \{1, 2, 3\}\}$ which is not normal, and let v be a game on $(X, \mathfrak{S}_1)$. Then we cannot know contributions of each single $\{2\}$ nor $\{3\}$. If we regard $\{\emptyset, \{1\}, \{1, 2, 3\}\}$ as the lattice, not the set system, the situation is a little different. In this case, the name of elements are just the label. We have $\mathcal{J}(\mathfrak{S}_1) = \{\{1\}, \{1, 2, 3\}\}$ and $|\mathcal{J}(\mathfrak{S}_1)| = 2$, so that considering $\mathfrak{S}_1$ as a set system $(\mathcal{J}(\mathfrak{S}_1), \mathfrak{S}_1)$, we can apply Definition 6 to the game on $\mathfrak{S}_1$, and we obtain $\phi^{\{1\}}(v)$ and $\phi^{\{1,2,3\}}(v)$.

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