Shapley value of capacities

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Abstract— **A definition of Shapley value for capacities and its axiomatization are proposed. This Shapley value encompasses the original Shapley valus and may have applicability to the capacity and game on set systems which satisfy the condition of normality.**

Keywords: Shapley value, multichoice game, set system

I. Introduction

The Shapley value is the most important value in the game theory which represents the average contribution of each element of the whole set [11]. We have proposed the generalized Shapley value using the concept of the maximal chain which can be applied to general capacities which includes bi-capacity and multichoice game, and so on [4]. This definition encompasses one of the original Shapley value.

To find suitable axiomatizations of the Shapley value is one of the important problems. There exist several studies for the axiomatization of Shapley value [7] [3]. There however exist little which are enough understandable or natural as one of the Shapley value. We propose another axiomatization of generalized Shapley value which is natural and understandable.

II. Set system and Shapley value of a capacity on it

In this section, we shall introduce some notations and definitions. Throughout this paper, we consider a finite universal set $X = \{1, 2, \ldots, n\}$, $n \geq 1$, and 2^X denotes the power set of X. Let us consider $\mathfrak{S}$ a subset of 2^X which contains $\emptyset$ and X. Then we call $(X, \mathfrak{S})$ (or simply $\mathfrak{S}$ if no confusion occurs) a *set system.*

Set systems are regarded as *partially ordered sets*, which has reflexivity, antisymmetry and transitivity, with ordering $\subseteq$.

Let $A, B \in \mathfrak{S}$. We say that A is *covered* by B, and write $A \prec B$ or $B \succ A$, if $A \subsetneq B$ and $A \subseteq C \subsetneq B$ together with $C \in \mathfrak{S}$ imply $C = A$.

Definition 1 (maximal chain of set system) Let $\mathfrak{S}$ be a set system. We call $\mathscr{C}$ a *maximal chain* of $\mathfrak{S}$ if $\mathscr{C} = (C_0, C_1, \ldots, C_m)$ satisfies $\emptyset = C_0 \prec C_1 \prec \cdots \prec C_m = X, C_i \in \mathfrak{S}, i = 0, \ldots, m.$

For a maximal chain $\mathscr{C} = (C_0, C_1, \ldots, C_m)$, the length of $\mathscr{C}$ is defined by m. We denote the set of all n-length maximal chains of $\mathfrak{S}$ by $\Gamma_n(\mathfrak{S})$ and all maximal chains of $\mathfrak{S}$ by $\Gamma(\mathfrak{S})$.

Definition 2 (totally ordered set system) We say that $(X, \mathfrak{S})$ is a *totally ordered set system* if for any $A, B \in \mathfrak{S}$, either $A \subseteq B$ or $A \supsetneq B$.

If $(X, \mathfrak{S})$ is a totally ordered set system, then $|\Gamma(\mathfrak{S})| = 1$.

Definition 3 (normal set system [4]) We say that $(X, \mathfrak{S})$ is a normal set system if for any $A \in \mathfrak{S}$, there exists at least one maximal chain $\mathscr{C} \in \Gamma_n(\mathfrak{S})$ satisfying $A \in \mathscr{C}$.

Example 4 We show examples of set systems in Fig. 1. All $\mathfrak{S}_1, \ldots, \mathfrak{S}_5$ are set systems. $\mathfrak{S}_1$, $\mathfrak{S}_2$, $\mathfrak{S}_3$ and $\mathfrak{S}_5$ are normal set systems, and $\mathfrak{S}_4$ is not a normal set system because there is no 3-length maximal chains which contain $\{3\}$. $\mathfrak{S}_3$ is a totally ordered set systems. $\mathfrak{S}_1$, $\mathfrak{S}_2$ and L_3 satisfies that all maximal chain is n-length, that is, 3-length maximal chains. $\mathfrak{S}_5$ has three maximal chains, $\{\emptyset, \{1\}, \{1,2\}, \{1,2,4\}, \{1,2,3,4\}$, $\{\emptyset, \{3\}, \{3,4\}, \{1,3,4\}, \{1,2,3,4\}\}$ and $\{\emptyset, \{1\}, \{1,3,4\}, \{1,2,3,4\}\}$. Among these, $\{\emptyset, \{1\}, \{1,2\}, \{1,2,4\}, \{1,2,3,4\}$ and $\{\emptyset, \{3\}, \{3,4\}, \{1,3,4\}, \{1,2,3,4\}\}$ are n-length maximal chains.

Definition 5 (capacity on a set system) Let $(X, \mathfrak{S})$ be a set system. A function $v : \mathfrak{S} \to [0,1]$ is a *capacity on* $(X, \mathfrak{S})$ if it satisfies $v(\emptyset) = 0, v(X) = 1$ and for any $A, B \in \mathfrak{S}, v(A) \leq v(B)$ whenever $A \subseteq B$.

Let v be a capacity on $(X, \mathfrak{S})$. For a $\mathscr{C} := (C_0, C_1, \ldots, C_m) \in \Gamma(\mathfrak{S})$, define $p^{v,\mathscr{C}}$ by

$$\begin{aligned} p^{v,\mathscr{C}} &:= (p_1^{v,\mathscr{C}}, p_2^{v,\mathscr{C}}, \ldots, p_m^{v,\mathscr{C}}) \\ &= (v(C_1) - v(C_0), v(C_2) - v(C_1), \\ &\qquad \ldots, v(C_m) - v(C_{m-1})), \end{aligned} \tag{1}$$

Note that $p^{v,\mathscr{C}}$ satisfies $p_i^{v,\mathscr{C}} \geq 0, i = 1, \ldots, m$ and $\sum_{i=1}^m p_i^{v,\mathscr{C}} = 1$.

Our propose definition of Shapley value is as follows.

Definition 6 (Shapley value of capacity) Let v be a capacity on a normal set system $(X, \mathfrak{S})$. The Shapley value of v, $\Phi(v) = (\phi_1(v), \ldots, \phi_n(v)) \in [0,1]^n$ is defined by

(GS) $$\phi_i(v) := \frac{1}{|\Gamma_n(\mathfrak{S})|} \sum_{\mathscr{C} \in \Gamma_n(\mathfrak{S})} (v(A_{\mathscr{C}} \cup \{i\}) - v(A_{\mathscr{C}})),$$

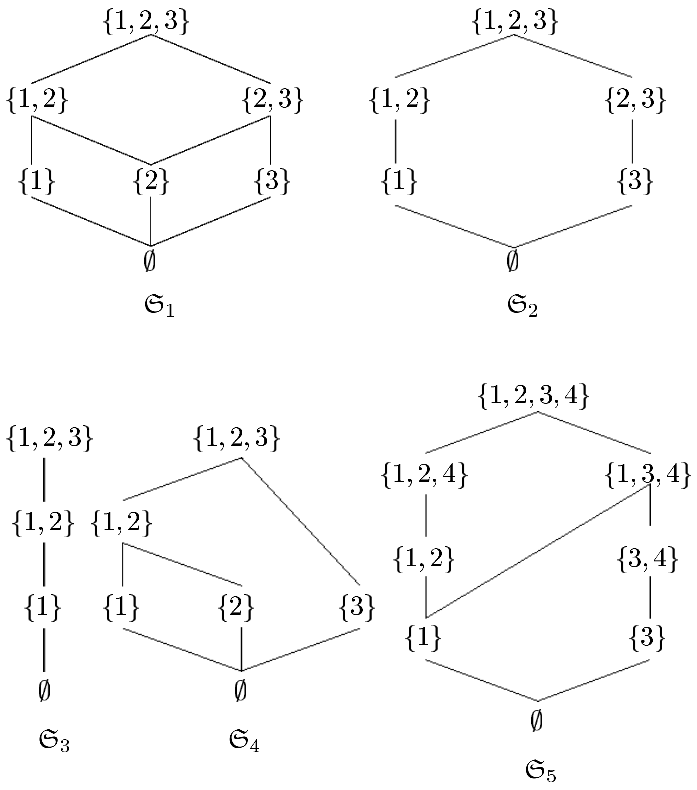

Fig. 1. set systems

where $A^i_{\mathscr{C}} := A \in \mathscr{C} \in \Gamma(\mathfrak{S})$ such that $A \not\ni \{i\}$ and $A \cup \{i\} \in \mathscr{C}$.

Remark that $\sum_{i=1}^n \phi_i(v) = 1$ holds.

Our definition (GS) encompass the original Shapley value, which is defined on a capacity on $(X, 2^X)$. The definition of the original Shapley value is as follows.

Definition 7 (Original Shapley value [11]) Let v be a capacity on $(X, 2^X)$. The Shapley value of v, $\Phi(v) = (\phi_1(v), \ldots, \phi_n(v)) \in [0,1]^n$ is defined by

$$\phi_i(v) := \sum_{A \subseteq X \setminus \{i\}} \gamma^n_{|A|}(v(A \cup \{i\}) - v(A)),\ i = 1, \ldots, n,$$

where

$$\gamma^n_k := \frac{(n-k-1)!k!}{n!}.$$

The Shapley value can be represented by using the maximal chains as follows.

Proposition 8 For any $\mathscr{C} \in \Gamma(2^X)$ and $i \in X$, there exists an $A_{\mathscr{C}} \in \mathscr{C}$ such that $\{i\} \notin A_{\mathscr{C}}$ and $A_{\mathscr{C}} \cup \{i\} \in \mathscr{C}$, and

$$\phi_i(v) = \frac{1}{n!} \sum_{\mathscr{C} \in \Gamma(2^X)} (v(A_{\mathscr{C}} \cup \{i\}) - v(A_{\mathscr{C}}))$$

holds.

This fact is well known in the game theory (cf. [6]). Our definition is based on the concept of above representation. Remark that $\Gamma(2^X) = \Gamma_n(2^X)$.

We introduce further concepts about capacities, which will be useful for stating axioms.

Definition 9 (dual capacity) Let v be a capacity on $(X, \mathfrak{S})$. Then the *dual capacity space* of v is defined on $\mathfrak{S}^d := \{A \in 2^X \mid A^c \in \mathfrak{S}\}$ by $v^d(A) := 1 - v(A^c)$ for any $A \in \mathfrak{S}^d$, where $A^c := X \setminus A$.

Definition 10 (permutation of v) Let π be a permutation on X. Then the *permutation* of v by π is defined on $\pi(\mathfrak{S}) := \{\pi(A) \in 2^X \mid A \in \mathfrak{S}\}$ by $\pi \circ v(A) := v(\pi^{-1}(A))$.

Let us consider a chain of length 2 as a set system, denoted by $\mathbf{2}$ (e.g., $\{\emptyset, \{1\}, \{1,2\}\}$), and a capacity $v^{\mathbf{2}}$ on it. We denote by the triplet $(0, u, 1)$ the values of $v^{\mathbf{2}}$ along the chain and we suppose $\mathbf{2} := \{\emptyset, \{1\}, \{1,2\}\}$ unless otherwise noted.

Definition 11 (embedding of $v^{\mathbf{2}}$) Let v be a capacity on a normal set system $(X, \mathfrak{S})$, and let $v^{\mathbf{2}} := (0, u, 1)$ be a capacity on $\mathbf{2}$. Then for $k \in X$, v^k is called the *embedding* of $v^{\mathbf{2}}$ into v at k and defined on the normal set system $(X^k, \mathfrak{S}^k)$ by

$$v^k(A) := \begin{cases} v(A^k) + u \cdot (v(A^k \cup \{k\}) - v(A^k)), & \text{if } A = A^k \cup \{k'\} \\ v(A), & \text{otherwise,} \end{cases} \tag{2}$$

where $A^k := A \in \mathfrak{S}$ such that $A \not\ni \{k\}$ and $A^k \cup \{k\} \in \mathfrak{S}$, $\{k\} = \{k', k''\}$, $X^k := \{1, \ldots, k-1, k', k'', k+1, \ldots, n\}$ and $\mathfrak{S}^k := \mathfrak{S} \cup \{A^k \cup \{k'\} \mid A^k \in \mathfrak{S}\}$.

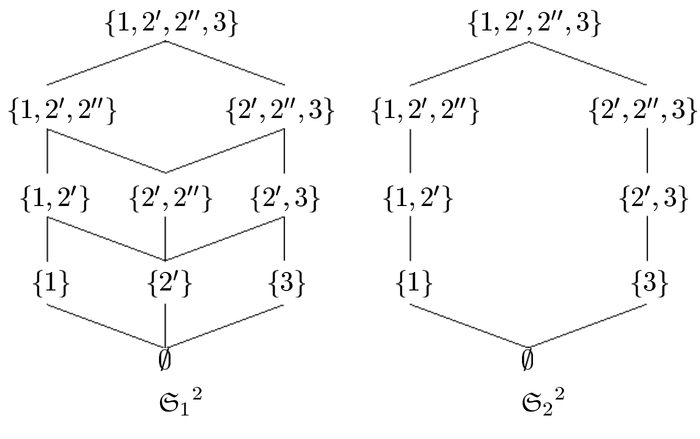

Fig. 2. set systems

Example 12 We give examples of embedding in Fig. 2. ${\mathfrak{S}_1}^2$ is the embedding of $\mathbf{2}$ into $\mathfrak{S}_1$ in Fig. 1 and ${\mathfrak{S}_2}^2$ is the embedding of $\mathbf{2}$ into $\mathfrak{S}_2$ in Fig. 1. Remark that $|\Gamma_3(\mathfrak{S}_1)| = 4 \neq |\Gamma_4({\mathfrak{S}_1}^2)| = 6$ and $|\Gamma_3(\mathfrak{S}_2)| = |\Gamma_4({\mathfrak{S}_2}^2)| = 2$.

We discuss about the domain of Φ. Let v be a capacity on $(X, \mathfrak{S})$. Then we call $(X, \mathfrak{S}, v)$ a capacity space. Let Σ_n

be the set of all normal set system of $X := \{1, 2, \ldots, n\}$ and let $\Delta_{\mathfrak{S}}$ be the set of all capacity space defined on normal set systems $(X, \mathfrak{S})$. The domains of Φ is $\Delta := \bigcup_{n=1}^{\infty} \bigcup_{\mathfrak{S} \in \Sigma_n} \Delta_{\mathfrak{S}}$, and Φ is a fuctions defined on $[0,1]^n$. We denote simply $\Phi(v)$ instead of $\Phi(X, \mathfrak{S}, v)$ as far as no confusion occurs. Remark that more properly, the dual capacity of v is the dual capacity space of the capacity space $(X, \mathfrak{S}, v)$ which is defined by $(X, \mathfrak{S}, v)^d := (X, \mathfrak{S}^d, v^d)$ with $\mathfrak{S}^d := \{A \in 2^X \mid A^c \in \mathfrak{S}\}$, the permutation of v is the permutation of the capacity space $(X, \mathfrak{S}, v)$ which is defined by $(X, \mathfrak{S}, v)^\pi := (X, \pi(\mathfrak{S}), \pi \circ v)$, and the embedding of $(0, u, 1)$ into v at k is the embedding of the capacity space $(\{1, 2\}, \mathbf{2}, (0, u, 1))$ into the capacity space $(X, \mathfrak{S}, v)$, and it is defined by $(X, \mathfrak{S}, v)^k := (X^k, \mathfrak{S}^k, v^k)$.

III. Axiomatization of the Shapley value of capacities

We introduce five axioms for the Shapley value of capacities.

(A1)(*continuity*) *For any* $(0, u, 1)$, $\phi_1(0, u, 1)$, *which can be regarded as a function with respect to* u *on* $(0, u, 1)$, *is continuous on* $[0, 1]$ *and there exists* u_0 *such that* $\phi_1(0, u_0, 1) > 0$.

(A2)(*dual invariance*) *For any* $(0, u, 1)$, $\Phi(0, u, 1) = \Phi(0, u, 1)^d$ *holds.*

(A3)(*embedding efficiency*) *Let* $(X, \mathfrak{S})$ *be a totally ordered normal set system. Then for any* v *on* $(X, \mathfrak{S})$, *any* $(0, u, 1)$ *and any* $k \in X$, $\phi_i(v^k) = \phi_i(v)$ *if* $i \neq k$, $\phi_{k'}(v^k) = \phi_k(v) \cdot \phi_1(0, u, 1)$ *and* $\phi_{k''}(v^k) = \phi_k(v) \cdot \phi_2(0, u, 1)$ *hold, where* $\{k\} = \{k', k''\}$.

(A4)(*convexity*) *Let* $(X, \mathfrak{S})$, $(X, \mathfrak{S}_1)$ *and* $(X, \mathfrak{S}_2)$ *be normal set systems satisfying* $\Gamma_n(\mathfrak{S}_1) \cup \Gamma_n(\mathfrak{S}_2) = \Gamma_n(\mathfrak{S})$ *and* $\Gamma_n(\mathfrak{S}_1) \cap \Gamma_n(\mathfrak{S}_2) = \emptyset$ *and* v *be a capacity on* $\mathfrak{S}$. *Then for any* $i = 1, \ldots, n$, *there exists an* $\alpha \in]0, 1[$ *such that* $\phi_i(v) = \alpha \phi_i(v|_{\mathfrak{S}_1}) + (1 - \alpha) \phi_i(v|_{\mathfrak{S}_2})$.

(A5)(*permutation invariance*) *Let* v *be a capacity on* 2^X. *Then for any permutaion* π *on* X, $\phi_i(v) = \phi_{\pi(i)}(\pi \circ v), i = 1, \ldots, n$ *holds.*

Then we obtain following theorem.

Theorem 13 ([6]) *Let* v *be a capacity on a normal set system* $(X := \{1\}, \{\emptyset, X\})$. *Under the condition* $\phi_1(v) = 1$, (GS) *holds if and only if* (A1), (A2), (A3), (A4) *and* (A5) *hold.*

When v is additive, $\phi_i(v) = v(\{i\})$ holds. Now we treat capacities defined on normal set systems. If the underlying space is not normal, Definition 6 can not be applied to such capacities. Because $\phi_i(v)$ is calculated as an avarage of i's contributions $v(A \cup \{i\} - A)$. For instance, let $X := \{1, 2, 3\}$. If v is defined on $\{\emptyset, \{1\}, \{1, 2, 3\}\}$ which is not normal, we can't know contributions of each single $\{2\}$ and $\{3\}$.

Now, We discuss in detail the above axioms.

A. Dual invariance

More generally, for any capacity on a normal set system, $\Phi(v)$ is dual invariant.

Proposition 14 For any capacity v on a normal set system, $\Phi(v^d) = \Phi(v)$.

Proof: Let v be a capacity on $\mathfrak{S}$. For any $A \in \mathfrak{S}$, $(A^c)^c = A$, hence the dual mapping is a bijection from $\mathfrak{S}$ to $\mathfrak{S}^d$. Then, $\mathscr{C} := (C_0, \ldots, C_n) \in \Gamma_n(\mathfrak{S})$ if and only if $\mathscr{C}^d := (C_n^c, \ldots, C_0^c) \in \Gamma_n(\mathfrak{S}^d)$, since $C_i \prec C_{i+1}$ implies $C_i^c \succ C_{i+1}^c$. Hence $|\Gamma_n(\mathfrak{S})| = |\Gamma_n(\mathfrak{S}^d)|$. Therefore

$$\begin{aligned}
&\phi_i(v^d) \\
&= \frac{1}{|\Gamma_n(\mathfrak{S}^d)|} \sum_{\mathscr{C} \in \Gamma_n(\mathfrak{S}^d)} \left(v^d(A_{\mathscr{C}} \cup \{i\}) - v^d(A_{\mathscr{C}})\right) \\
&= \frac{1}{|\Gamma_n(\mathfrak{S}^d)|} \sum_{\mathscr{C} \in \Gamma_n(\mathfrak{S}^d)} \left((1 - v^d(A_{\mathscr{C}})) - (1 - v^d(A_{\mathscr{C}} \cup \{i\}))\right) \\
&= \frac{1}{|\Gamma_n(\mathfrak{S})|} \sum_{\mathscr{C} \in \Gamma_n(\mathfrak{S})} (v(B \cup \{i\}) - v(B)) = \phi_i(v),
\end{aligned}$$

where $A_{\mathscr{C}} := A \in \mathscr{C} \in \Gamma_n(\mathfrak{S})$ such that $\{i\} \notin A$ and $A \cup \{i\} \in \mathscr{C}$, and $B := X \setminus (A_{\mathscr{C}} \cup \{i\})$. ■

B. Embedding efficiency

Let v be a capacity on a totally ordered set system $\mathfrak{S} := \{A_0, \ldots, A_n\}$ such that $A_{j-1} \prec A_j$, $j = 1, \ldots, n$, Then the embedding at k means that $\{k\}$ is splitted to $\{k', k''\}$. In fact,

$$v^k(A) := \begin{cases} v(A^k) + u(v(A^k \cup \{i\}) - v(A^k)), & \\ & \text{if } A = A^k \cup \{k'\} \\ v(A), & \text{otherwise,} \end{cases}$$

is defined on $(X^k, \mathfrak{S}^k)$, where $X^k := (1, \ldots, k-1, k', k'', k+1, \ldots, n\}$, $\mathfrak{S}^k := \{A_0, \ldots, A^k, A^k \cup \{k'\}, A^k \cup \{k\}, \ldots, A_n\}$ such that $A_{j-1} \prec A_j$, $j = 1, \ldots, n$ and $A^k \prec A^k \cup \{k'\} \prec A^k \cup \{k\}$.

(A3) implies $\phi_{k'}(v^k) + \phi_{k''}(v^k) = \phi_k(v)$ and $\phi_{k'}(v^k) / \phi_{k''}(v^k) = u/(1 - u)$ so that (A3) is a natural axiom in the meaning of the contributions of k' and k''.

More generally, this fact holds for any capacity on a normal set system.

Proposition 15 Let v be a capacity on $(X, \mathfrak{S})$. If for $k \in X$, $|\Gamma_n(\mathfrak{S})| = |\Gamma_{n+1}(\mathfrak{S}^k)|$, then $\phi_{k'}(v^k) = \phi_k(v) \cdot \phi_1(0, u, 1)$ and $\phi_{k''}(v^k) = \phi_k(v) \cdot \phi_2(0, u, 1)$ hold.

C. Permutation invariance

More generally, even in the case of that v is defined on normal set systems which is not 2^X, $\Phi(v)$ is permutation invariant.

Proposition 16 Let v be a capacity on a normal set system. Then for any permutation on X, $\phi_{\pi(i)}(\pi \circ v) = \phi_i(v)$.

Proof: For any $\mathscr{C} = (C_0, C_1, \dots, C_m) \in \Gamma_n(\pi(\mathfrak{S}))$, $\pi^{-1}(\mathscr{C}) := (\pi^{-1}(C_0), \pi^{-1}(C_1), \dots, \pi^{-1}(C_m)) \in \Gamma_n(\mathfrak{S})$, because for any $A, B \in \pi(\mathfrak{S})$, $A \prec B$ implies $\pi^{-1}(A) \prec \pi^{-1}(B)$, so that we have for $i = 1, \dots, n$,

$$\begin{aligned}
&\phi_{\pi(i)}(\pi \circ v) \\
&= \frac{1}{|\Gamma_n(\pi(\mathfrak{S}))|} \sum_{\mathscr{C} \in \Gamma_n(\pi(\mathfrak{S}))} (\pi \circ v(A_{\mathscr{C},\pi(\{i\})} \cup \pi(\{i\})) \\
&\qquad - \pi \circ v(A_{\mathscr{C},\pi(\{i\})})) \\
&= \frac{1}{|\Gamma_n(\mathfrak{S})|} \sum_{\mathscr{C} \in \Gamma_n(\mathfrak{S})} (v(\pi^{-1}(A_{\mathscr{C}}) \cup \{i\}) - v(\pi^{-1}(A_{\mathscr{C}}))) \\
&= \frac{1}{|\Gamma_n(\mathfrak{S})|} \sum_{\mathscr{C} \in \Gamma_n(\mathfrak{S})} (v(A_{\mathscr{C}} \cup \{i\}) - v(A_{\mathscr{C}})) = \phi_i(v),
\end{aligned}$$

where $A_{\mathscr{C},\{j\}} := A \in \mathscr{C} \in \Gamma_n(\mathfrak{S})$ such that $\{j\} \notin A$ and $A \cup \{j\} \in \mathscr{C}$. ■

Here we show three lemmas used in the proof of Theorem 13.

Lemma 17 ([6]) Let $f(x)$ be a continuous function on $\mathbb{R}$.
(i) For any x, y, $f(x+y) = f(x) + f(y)$ holds if and only if $f(x) = \alpha x, \alpha \in \mathbb{R}$.
(ii) For any x, y, $f(x+y) = f(x)f(y)$ holds if and only if $f(x) = e^{\alpha x}, \alpha \in \mathbb{R}$, or constant valued $f(x) = 0$.
(iii) For any x, y, $f(xy) = f(x)f(y)$ holds if and only if $f(x) = x^\alpha, \alpha \in \mathbb{R}$.

Lemma 18 ([5]) Let $(X, \mathfrak{S})$ be a normal set system which is not totally ordered. Then there exist two normal set systems such that $\mathfrak{S} = \mathfrak{S}_1 \cup \mathfrak{S}_2$, $\Gamma_n(\mathfrak{S}_1) \cup \Gamma_n(\mathfrak{S}_2) = \Gamma_n(\mathfrak{S})$ and $\Gamma_n(\mathfrak{S}_1) \cap \Gamma_n(\mathfrak{S}_2) = \emptyset$.

Lemma 19 ([5]) For any normal set systems $\mathfrak{S} \subsetneq 2^X$, there exists a normal set system $\mathfrak{S}'$ such that $\mathfrak{S} \cup \mathfrak{S}' = 2^X$, $\Gamma_n(\mathfrak{S}) \cup \Gamma_n(\mathfrak{S}') = 2^X$ and $\Gamma_n(\mathfrak{S}) \cap \Gamma_n(\mathfrak{S}') = \emptyset$.

IV. Application to capacity on lattice

The *lattice* $(L, \leq)$ is a partially ordered set such that for any pair $x, y \in L$, there exist a least upper bound $x \vee y$ (supremum) and a greatest lower bound $x \wedge y$ (infimum) in L. Consequently, for finite lattices, there always exist a greatest element (supremum of all elements) and a least element (infimum of all elements), denoted by $\top, \bot$ (see [1]). Our approach may have applicability to capacities defind on lattices which satisfy a kind of normality by the translation from lattices to set systems (cf. [4]).

Definition 20 (capacity on lattice) Let $(L, \leq)$ be a finite lattice with greatest and least elements denoted by $\top$ and $\bot$ respectively. A *capacity* on L is a function $v : L \to [0,1]$ satisfying $v(\bot) = 0$, $v(\top) = 1$, and being isotone, i.e., $x \leq y$ implies $v(x) \leq v(y)$.

Evidently a set system is not necessarily a lattice. Moreover, a normal set system is not necessarily a lattice. Indeed, take $X = \{1,2,3,4\}$ and $\mathfrak{S} := \{\emptyset, \{1\}, \{3\}, \{1,2\}, \{2,3\}, \{1,4\}, \{3,4\}, \{1,2,3\}, \{1,3,4\}, X\}$. Then, $\{1\}$ and $\{3\}$ have no supremum.

Definition 21 (join-irreducible element) An element $x \in (L, \leq)$ is *join-irreducible* if for all $a, b \in L$, $x \neq \bot$ and $x = a \vee b$ implies $x = a$ or $x = b$.

We denote the set of all join-irreducible elements of L by $\mathcal{J}(L)$. Similarly, meet-irreducible elements are defined by replacing $\vee$ by $\wedge$ in the above definition. The set of all meet-irreducible elements is denoted by $\mathcal{M}(L)$.

The mapping η for any $a \in L$, defined by

$$\eta(a) := \{x \in \mathcal{J}(L) \mid x \leq a\}$$

is a lattice-isomorphism of L onto $\eta(L) := \{\eta(a) \mid a \in L\}$, that is, $(L, \leq) \cong (\eta(L), \subseteq)$. (See Section V-A)

Translating lattices which is underlying space of a capacity v to set systems, we can apply Definition 6 and also Definition 7 to a capacity on them if that is the case where the set system is normal.

If we regard $\mathfrak{S} := \{\emptyset, \{1\}, \{1,2,3\}\}$ as the lattice, not the set system, the situation is a little different. In this case, the name of elements are just the label. We have $\mathcal{J}(\mathfrak{S}) = \{\{1\}, \{1,2,3\}\}$ and $|\mathcal{J}(\mathfrak{S})| = 2$, so that considering $\mathfrak{S}$ as a lattice, we can apply Definition 7 and 6 to $\mathfrak{S}$.

V. Examples

In this section, we show several examples. Most games and capacities which appear in applications are particular capacities on normal set systems.

A. Regular lattice

L in Fig. 3 is a lattice.

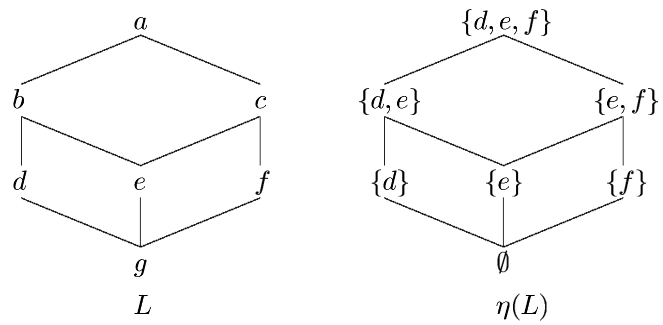

Fig. 3.

$\mathcal{J}(L_1) = \{d, e, f\}$, and L_1 is also represented by $\eta(L_1)$. We have $|\mathcal{J}(L_1)| = 3$, then $\Gamma_3(\eta(L_1)) = \{(\emptyset, \mathrm{d}, \mathrm{de}, \mathrm{def}),$

$(\emptyset, \mathrm{e}, \mathrm{de}, \mathrm{def})$, $(\emptyset, \mathrm{e}, \mathrm{ef}, \mathrm{def}), (\emptyset, \mathrm{f}, \mathrm{ef}, \mathrm{def})\}$. Let v be a capacity on L_1. Then the Shapley values are as follows.

$$\begin{aligned}
&\phi_d(v)\\
&= \frac{1}{4}(v(d)-v(g)) + \frac{1}{4}(v(b)-v(e)) + \frac{1}{2}(v(a)-v(c))\\
&\phi_e(v)\\
&= \frac{1}{2}(v(e)-v(g)) + \frac{1}{4}(v(b)-v(d)) + \frac{1}{4}(v(c)-v(f))\\
&\phi_f(v)\\
&= \frac{1}{4}(v(f)-v(g)) + \frac{1}{4}(v(c)-v(e)) + \frac{1}{2}(v(a)-v(b)).
\end{aligned}$$

B. Bi-capacity

A bi-capacity is a monotone function on $\mathcal{Q}(X) := \{(A,B) \in 2^X \times 2^X \mid A \cap B = \emptyset\}$ which satisfies that $v(\emptyset, X) = -1, v(\emptyset,\emptyset) = 0$ and $v(X,\emptyset) = 1$. For any $(A_1,A_2),(B_1,B_2) \in \mathcal{Q}(X)$, $(A_1,A_2) \sqsubseteq (B_1,B_2)$ iff $A_1 \subseteq B_1$ and $A_2 \supseteq B_2$. $\mathcal{Q}(X) \cong 3^X$. It can be shown that $(\mathcal{Q}(X), \sqsubseteq)$ is a finite distributive lattice. Sup and inf are given by $(A_1,A_2) \vee (B_1,B_2) = (A_1 \cup B_1, A_2 \cap B_2)$ and $(A_1,A_2) \wedge (B_1,B_2) = (A_1 \cap B_1, A_2 \cup B_2)$, and we have

$$\mathcal{J}(\mathcal{Q}(X)) = \{(\emptyset, X\setminus\{i\}), i \in X\} \cup \{(\{i\}, X\setminus\{i\}), i \in X\},$$

where $i \in X$. Normalizing v by $v' : \mathcal{Q}(X) \to [0,1]$ such that

$$v' := \frac{1}{2}v + \frac{1}{2},$$

we can regard v as a capacity on $\mathcal{Q}(X)$. Then, applying (GS), we have

$$\begin{aligned}
&\phi_i^+(v')\\
&:= \phi_{(\{i\},X\setminus\{i\})}(v')\\
&= \sum_{\substack{A\subseteq X\setminus\{i\}\\ B\subseteq X\setminus(A\cup\{i\})}} \gamma^n_{|A|,|B|}\left(v'(A\cup\{i\},B) - v'(A,B)\right),\\
&\phi_i^-(v')\\
&:= \phi_{(\emptyset,X\setminus\{i\})}(v')\\
&= \sum_{\substack{A\subseteq X\setminus\{i\}\\ B\subseteq X\setminus(A\cup\{i\})}} \gamma^n_{|A|,|B|}\left(v'(B,A) - v'(B,A\cup\{i\})\right).
\end{aligned}$$

where

$$\gamma^n_{k,\ell} := \frac{(n-k+\ell-1)!\,(n+k-\ell)!\,2^{n-k-\ell}}{(2n)!}.$$

ϕ_i^+ and ϕ_i^- mean positive and negative degrees of i's contribution to v, respectively, hence the contribution of i to v is given by $\phi_i(v) := \phi_i^+(v) + \phi_i^-(v)$. $\gamma^n_{|A|,|B|}$ is the rate of the number of chains which contain $(A\cup\{i\},B)$ and (A,B). In fact,

$$\begin{aligned}
&|\{C \in \Gamma_{2n}(\mathcal{Q}(X)) \mid C \ni (A\cup\{i\},B),(A,B)\}|\\
&\quad= \frac{(n+|A|-|B|)!}{(2!)^{|A|}} \cdot \frac{(n-|A|+|B|-1)!}{(2!)^{|B|}}
\end{aligned}$$

and $|\Gamma_{2n}(\mathcal{Q}(X))| = (2n)!/(2!)^n$.

C. Multichoice game

Multichoice games have been proposed by Hsiao and Raghavan [7].

Let $X := \{0,1,\ldots .n\}$ be a set of players, and let $L := L_1 \times \cdots \times L_n$, where $(L_i, \leq_i)$ is a totally ordered set $L_i = \{0,1,\ldots,\ell_i\}$ such that $0 \leq_i 1 \leq_i \cdots \leq_i \ell_i$. Each L_i is the set of choices of player i. $(L, \leq)$ is a normal lattice. For any $(a_1,a_2,\ldots,a_n),(b_1,b_2,\ldots,b_n) \in L$, $(a_1,a_2,\ldots,a_n) \leq (b_1,b_2,\ldots,b_n)$ iff $a_i \leq_i b_i$ for all $i = 1,\ldots,n$. We have

$$\mathcal{J}(L) = \{(0,\ldots,0,a_i,0,\ldots,0) \mid a_i \in \mathcal{J}(L_i) = L_i \setminus \{0\}\}$$

and $|\mathcal{J}(L)| = \sum_{i=1}^n \ell_i$. The lattice in Fig. 4 is an example of a product lattice, which represents a 2-players game. Players 1 and 2 can choose among 3 and 4 choices. Let v be

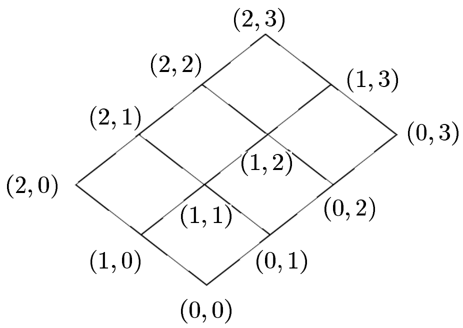

Fig. 4. 2-players game

a capacity on L, that is, $v(0,\ldots,0) = 0$, $v(\ell_1,\ldots,\ell_n) = 1$ and , for any $a,b \in L$, $v(a) \leqq v(b)$ whenever $a \leq b$. In this case, applying (GS), we have

$$\begin{aligned}
\phi_i^j(v) &= \phi_{(0,\ldots,0,a_i=j>0,0,\ldots,0)}(v)\\
&= \sum_{a\in L/L_i} \xi_i^{(a,j)}\left(v(a,j) - v(a,j-1)\right),
\end{aligned}$$

where $L/L_i := L_1 \times \cdots \times L_{i-1} \times L_{i+1} \times \cdots \times L_n$, $(a,a_i) := (a_1,\ldots,a_{i-1},a_i,a_{i+1},\ldots,a_n) \in L$ such that $a \in L/L_i$ and $a_i \in L_i$, and

$$\xi_i^{(a,a_i)} := \left(\prod_{k=1}^n \binom{\ell_k}{a_k}\right) \cdot \binom{\sum_{k=1}^n \ell_k}{\sum_{k=1}^n a_k}^{-1} \cdot \frac{a_i}{\sum_{k=1}^n a_k}.$$

$\phi_i^j(v)$ represents the contribution of player i playing at level j compared to level $j-1$, where $j, j-1 \in \mathcal{J}(L_i) = L_i \setminus \{0\}$, hence player i's overall contribution is given by

$$\phi_i(v) = \sum_{j=1}^{\ell_i} \phi_i^j(v).$$

$\xi_i^{(a,a_i)}$ is the rate of the number of chains which contain (a,a_i) and (a,a_i-1) among $(\sum_{i=1}^n \ell_i)$-length maximal chain. In fact, put $m := (\sum_{i=1}^n \ell_i)$, then

$$\begin{aligned}
&|\{\mathscr{C} \in \Gamma_m(L) \mid \mathscr{C} \ni (a,a_i), \mathscr{C} \ni (a,a_i-1)\}|\\
&\quad= \frac{(\sum_{k=1}^n a_k - 1)!}{(\prod_{k=1}^n (a_k!))(a_i-1)!/(a_i!)} \cdot \frac{(\sum_{k=1}^n (\ell_k - a_k))!}{\prod_{k=1}^n ((\ell_k-a_k)!)}
\end{aligned}$$

and $|\Gamma_m(L)| = (\sum_{k=1}^{n} \ell_k)!/\prod_{k=1}^{n}(\ell_k!)$.

Regarding a bi-capacity in Section V-B as a special case of multichoice game such that n players and $\ell_i = 2$ for all i which is fixed a value $v'(\emptyset, \emptyset) = 1/2$, we obtain the same Shapley values.

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