Distorted probability generated by triangular norm

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Abstract— **A new class of fuzzy measure generated by triangular norm presented by 2 parameter, which is called (λ, p)-fuzzy measure, is proposed. It is a distorted probability and also a generalization of λ-fuzzy measure. Consequently, it is readily-treatable and cover wide range of fuzzy measures, and it also encompasses possibility measure. We propose also the method of identification of fuzzy measures using proposed class.**

Keywords: λ-fuzzy measure, (λ, p)-fuzzy measure, distorted probability, subjective evaluation

I. Introduction

Fuzzy measures, which is nonadditive, can be used to express interactions between items that are not expressible by additive measure. Several studies are being done on their application in variety of field such as analysis of subjective evaluation problem, decision-making support and pattern recognition.

When data is collected as fuzzy measures in the real world, fuzzy measure are rarely all given as data, making it necessary to estimate the measures of whole events from partial known data. This is called the problem of identifying fuzzy measures. To identify fuzzy measures, they can be restricted to a certain class, and methods using λ-fuzzy measure, χ-fuzzy measure and distorted probability have been proposed [1, 4–6]. These each methods however have some problems.

In the methods using λ-fuzzy measure or χ-fuzzy measure, we have to only estimate a parameter λ or χ. It is simple, but we can not express a multiplier effect and cancel effect, in other words, superadditive and subadditive at a same time using one parameter λ or χ. In the methods using distorted probability, we can express a wide range of fuzzy measures which is also even superadditive in some parts and subadditive in some parts at a same time. But to identify using distorted probability, we have to estimate a combination of a probability measure and a scaling function. It is vexatious complication.

In this paper, we propose (λ, p)-fuzzy measure which has two parameters. This fuzzy measure can be express superadditive and subadditive at a same time with one pair of the parameter (λ, p). It also contains λ-fuzzy measures and possibility measures. We propose an identifying fuzzy measures using a classified class of (λ, p)-fuzzy measures, and verify its feasibility. Moreover we discuss on fuzzy measures generated by triangular norm.

II. Preliminaries

In this section, we introduce some definitions. Throughout this paper, we consider a finite universal set $X := \{1, 2, \dots, n\}$, $n > 1$, and 2^X denote the power set of X.

Definition 1 If a set function $v : 2^X \to [0, 1]$ satisfies $v(\emptyset) = 0, v(X) = 1$ and for any $A, B \in 2^X$, $v(A) \leq v(B)$ whenever $A \subseteq B$, v is calld a *fuzzy measure.*

Definition 2 If a fuzzy measure v on 2^X satisfies that for any $A, B \in 2^X$ satisfying $A \cap B = \emptyset$,

$$v(A \cup B) = v(A) + v(B) + \lambda v(A)v(B), \quad -1 < \lambda < \infty,$$

then v is called a λ-*fuzzy measure.*

Definition 3 ([7]) If for fuzzy measure v on 2^X there exists a χ such that

$$\chi \geq 1 - \frac{\min_{i \in X} v(\{i\})}{\sum_{i \in X} v(\{i\})},$$

and

$$v(A) = \chi^{|A|-1} \left(\sum_{i \in A} v(\{i\}) \right)$$

holds for any $A \in 2^X$, where $|A|$ is the number of elements of A, then v is called a χ-*fuzzy measure.*

Definition 4 If a fuzzy measure v on 2^X satisfies that for any $A, B \in 2^X$,

$$v(A \cup B) = \max\{v(A), v(B)\},$$

then v is called a *possibility measure.*

Definition 5 If a set function $P : 2^X \to [0, 1]$ satisfies that for any $A, B \in 2^X$ satisfying $A \cap B = \emptyset$,

$$P(A \cup B) = P(A) + P(B),$$

then P is called a *probability measure.*

Definition 6 ([4], cf. Sec. 2.2.9 of [8] and Sec. 3.3 of [11]) If for a fuzzy measure v on 2^X, there exist a probability measure P and a nondecreasing function $f : [0, 1] \to [0, 1]$ such that $v(A) = f \circ P(A)$ for any $A \in 2^X$, then v is called a *distorted probability.*

III. (λ, p)-FUZZY MEASURE

In this section, we propose (λ, p)-fuzzy measure and show its properties.

Definition 7 Let v be a fuzzy measure on $(X, 2^X)$. If v satisfies that for any $A, B \in 2^X$ satisfying $A \cap B = \emptyset$,

$$v(A \cup B) = \sqrt[p]{v(A)^p + v(B)^p + \lambda v(A)^p v(B)^p},$$

$0 < p \leq \infty$ and $-1 < \lambda < \infty$, then v is called a (λ, p)-fuzzy measure.

Proposition 8 The λ-fuzzy measure is a (λ, p)-fuzzy measure with $p = 1$. The possibility measure is a (λ, p)-fuzzy measure with $p = \infty$.

Proposition 9 The (λ, p)-fuzzy measure is distorted probability.

Proof Putting $P(A) := \log_{(1+\lambda)}(1 + \lambda v(A)^p)$ and

$$f(x) := \sqrt[p]{\frac{(1+\lambda)^x - 1}{\lambda}},$$

we have $v(A) = f \circ P(A)$ for any $A \in 2^X$.

We show graphs of $P(A)$, $\lambda = 2, p = 0.5, 1, 2$ and 3 (Fig. 1), and $f(x)$, $\lambda = 3, p = 1000$ (2). As Fig 2 shows, the graph of $f(x), \lambda = 3, p = 2$ is both convex at some interval and concave at some interval.

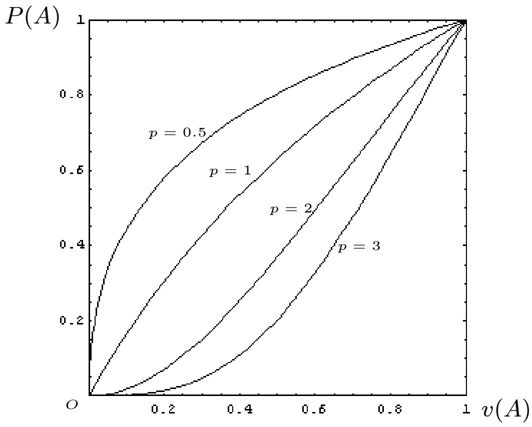

Fig. 1. Graph of $P(A), \lambda = 2$

Proposition 10 Let v be a (λ, p)-fuzzy measure. v can be both superadditive for some $A, B \in 2^X$ and subadditive for some $A, B \in 2^X$.

Proof For example, putting $p = 2$, we have $v(A \cup B) \geq v(A) + v(B)$ if $\lambda v(A)v(B) \geq 2$ and $v(A \cup B) \leq v(A) + v(B)$ if $\lambda v(A)v(B) \leq 2$.

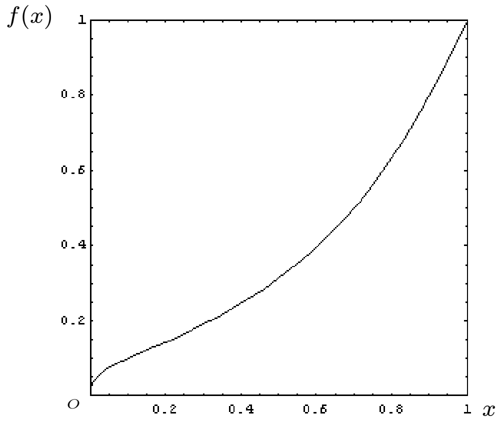

Fig. 2. Graph of $f(x), \lambda = 3, p = 1000$

IV. GENERATE DISTORTED PROBABILITY BY TRIANGULAR NORMS

In this section, we discuss about the generation of the fuzzy measure using triangular norm.

Definition 11 If a operation $T : [0,1] \times [0,1] \to [0,1]$ satisfies

(1) $T(0,0) = 0, T(x,1) = x$ for any $x > 0$,
(2) $x \leq y$ implies $T(x,z) \leq T(y,z)$,
(3) $T(x,y) = T(y,x)$ and
(4) $T(x, T(y,z)) = T(T(x,y), z)$,

then T is called triangular norm.

Definition 12 If a operation $S : [0,1] \times [0,1] \to [0,1]$ satisfies

(1') $S(1,1) = 1, S(x,0) = x$ for any $x < 1$,

(2), (3) and (4), then S is called triangular conorm.

For a triangular norm T, $S(x,y) := 1 - T(1-x, 1-y)$ is a triangular conorm and called a dual of T.

Proposition 13 ([9]) If a triangular norm T satisfiles that $x \to a$ implies $T(x,y) \to T(a,y)$ for any $y \in [0,1]$ and that $y < z$ implies $T(x,y) < T(x,z)$ for any $x \in [0,1]$, then there exists some functions $\varphi : [0,1] \to [0,1]$ such that for any $x, y \in [0,1]$,

$$T(x,y) = \varphi^{-1}(\varphi(x)\varphi(y))$$

holds.

$\varphi(x)$ is called a multiplicative generating function of a triangular norm. Using by generating function $\varphi(x)$, we can also generate triangular conorms .

Proposition 14 Let $\varphi(x)$ be a multiplicatively generating function of a triangular norm T and S be the dual of T. Then

$$S(x,y) = \psi^{-1}(\psi(x)\psi(y)),$$

where

$$\psi(x) := \varphi(1-x),$$

is a triangular conorm and dual of T.

Proposition 13 and 14 means that we can generate triangular norms and conorms choosing appropriate generating functions.

It is well known that triangular conorms are concerned with fuzzy measures [2]. We can generate distorted probability using the similar way of the above.

Proposition 15 If for a fuzzy measure v, there exists an $\eta(x)$ which is a nonincreasing or a nondecreasing function on $[0,1]$, $\eta(0)=1$ and $\eta(1)>0$ such that

$$v(A\cup B)=\eta^{-1}(\eta(x)\eta(y)),$$

where $x:=v(A), y:=v(B)$, then v is a distorted probability.

Proof We have $v(A\cup B)=S(v(A),v(B))=\eta^{-1}((\psi\circ v(A))(\psi\circ v(B)))$, which implies $\eta\circ v(A\cup B)=(\eta\circ v(A))(\eta\circ v(B))$. Putting $b:=\eta(1)$ and $P(A):=\log_b\circ\eta\circ v(A)$, we have $P(A\cup B)=P(A)+P(B)$ which means P is a probability measure and we can represent

$$v(A)=(\log_b\circ\eta)^{-1}\circ P(A)=f\circ P(A),$$

where $f(x):=(\log_b\circ\eta)^{-1}(x)$. And we have $f(0)=0$, $f(1)=1$ and $f'(x)\geq 0$, which imply v is a distorted probability.

In the case of distorted probabilities, differently from triangular norms and conorms, η is not necessary $[0,1]\to[0,1]$. Using Proposition 15, we can generate fuzzy measure which is distorted probability.

Example 16 we obtain (λ,p)-fuzzy measure which is a distorted probability.

Put $\eta(x):=\lambda x^p+1$, $0<p,\lambda>-1$. We have $\eta(0)=1$, $\eta(1)=\lambda+1>0$ and $\eta(x)$ is a a nonincreasing or a nondecreasing function on $[0,1]$ since $\eta'(x)=\lambda p x^{p-1}$. Then we obtain

$$\begin{aligned}\eta^{-1}(\eta(x)\eta(y)) &= \eta^{-1}(\eta(x)\eta(y))\\ &= \sqrt[p]{x^p+y^p+\lambda x^p y^p}.\end{aligned}$$

Therefore a fuzzy measure which satisfies

$$v(A\cup B)=\sqrt[p]{v(A)^p+v(B)^p+\lambda v(A)^p v(B)^p},$$

$\lambda>-1, 0<0\leq\infty$ is a distorted probability, and we obtain $P(A):=\log_{\lambda+1}(\lambda v(A)^p+1)$ and

$$f(x):=\eta^{-1}\circ(\log_{1+\lambda})^{-1}(x)=\sqrt[p]{\frac{(\lambda+1)^x-1}{\lambda}}.$$

Example 17 Put $\eta(x):=1+\dfrac{\lambda x}{1+x}$, $-\frac{3}{2}<\lambda<1,\lambda\neq 0$. We have $\eta(0)=1$, $\eta(1)=1+\frac{\lambda}{2}>0$ and $\eta(x)$ is a a nonincreasing or a nondecreasing function on $[0,1]$ since $\eta'(x)=\frac{\lambda}{1+x^2}$. Then we obtain

$$\begin{aligned}\eta^{-1}(\eta(x)\eta(y)) &= \eta^{-1}(\eta(x)\eta(y))\\ &= \frac{x+y+2xy+\lambda xy}{1-xy-\lambda xy}.\end{aligned}$$

Therefore a fuzzy measure which satisfies

$$v(A\cup B)=\frac{v(A)+v(B)+2v(A)v(B)+\lambda v(A)v(B)}{1-v(A)v(B)-\lambda v(A)v(B)},$$

$\lambda>-2$ is a distorted probability, and we obtain $P(A):=\log_{1+\frac{\lambda}{2}}\left(1+\frac{\lambda v(A)}{1+v(A)}\right)$ and

$$f(x):=\frac{\left(1+\frac{\lambda}{2}\right)^x-1}{\lambda-\left(1+\frac{\lambda}{2}\right)^x+1}.$$

V. Identification of fuzzy measures

In this section, we verify the feasibility of the identifying using (λ,p)-fuzzy measure by comparing other methods. Let $X:=\{1,\ldots,n\}$ be a set of items. Suppose that the fuzzy measures of partial events $\mathcal{K}\subsetneq 2^X$ are given, and that it has the value $v(A), A\in\mathcal{K}$. We wish to minimize squared errors of identification results relative to true values (least-squares estimation).

Identifying fuzzy measures using (λ,p)-, λ- or χ-fuzzy measure means estimating parameters (λ,p), λ or χ. Thus, let $\bar{v}$ be a estimation of v, then the parameters that minimizes

$$\sum_{A\in\mathcal{K}}(v(A)-\bar{v}(A))^2.$$

To determin parameters we use Newton method.

If $\mathcal{K}$ does not contain all singletons in which case, estimation is still possible, λ is estimated and used to determine measures of unknown singletons, which are used along with the estimated value of parameters for identification.

A. Data used

Data was collected using a questionnaire as follows:

The subject was asked to name some product s/he is interested in obtaining, along with four evaluation items s/he would consider when making a selection. The subject was then asked to rate the degree to which s/he would likely purchase the product when only one item, two items, three items, and all four items were satisfactorily provided. A personal computer, for example, may have the four evaluation items of design, functions, stability and brand. When the computer only has a high degree of design, it means that the other items are not provided satisfactorily.

Subjects were asked to give their ratings.

The four items are denoted $1,2,3$ and 4 and normalized so the $v(X)=1$. It was assumed that $v(\{1\}),v(\{2\}),v(\{3\}),v(\{4\}),v(\{1,2\}),v(\{3,4\}),v(\{1,2,3\})$ and $v(X)$ are known; there were used to identify remaining data values. The unknown data consist of $v(\{1,3\}),v(\{1,4\}),v(\{2,3\}),v(\{2,4\}),v(\{1,2,4\}),v(\{1,3,4\})$ and $v(\{2,3,4\})$.

B. Results and discussion

Results of identification obtained from the three methods and the means of the sums of squared errors are presented in Table I. Squared errors when known data was replaced by identification results (Err_1) and those when original known data was used (Err_2) are both presented.

$$Err_1 = \frac{1}{20}\sum_{j=1}^{20}\sum_{A\in 2^X}(v(A)-\bar{v}(A))^2$$

and

$$Err_2 = \frac{1}{20}\sum_{j=1}^{20}\sum_{A\in 2^X\setminus\mathcal{K}}(v(A)-\bar{v}(A))^2,$$

where $\bar{v}$ is the estimated value of v.

TABLE I

Squared errors of estimated values

Estimation	λ-fuzzy	χ-fuzzy	(λ, p)-fuzzy
Err_1	0.0922	0.0874	0.0434
Err_2	0.0650	0.0602	0.0337

While fairly satisfactory identification results were produced with all of the methods tested in the present case study, the method using (λ, p)-fuzzy measure produced the best results. In term of the number of best identification based on squared error Err_1 among the 20 data samples, 14 were by (λ, p)-fuzzy measures, 4 were by λ-fuzzy measures and 2 were by χ-fuzzy measures. When based on squared error Err_2, we obtained the result same as Err_1.

Our proposal produced superior results as compared to previous conventional methods. It is natural because (λ, p)-fuzzy measure is expressed by two parameters concerning λ- and χ-fuzzy measures expressed by one parameter. It is, therefore, can cover a more wide range of fuzzy measures. Despite this, (λ, p)-fuzzy measure is treatable as well as the others. We can say (λ, p)-fuzzy measure express the subjective decision making better than the other method.

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