Subjective Evaluation Process by
Multi-Dimensional Fuzzy Measure

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I. INTRODUCTION

The measure of subjective evaluation is formulated as a fuzzy measure on the set of items \( X = \{ x_1, x_2, \ldots, x_k \} \).
A fuzzy measure is a set function \( \mu : 2^X \to [0, 1] \) satisfying

\[
\mu(\emptyset) = 0, \mu(X) = 1, \quad \text{if } A \subset B, A, B, \in 2^X, \text{ then } \mu(A) \leq \mu(B)
\]

(monotonicity).

If an object has a score function \( f(x) : X \to [0, 1] \), then its evaluated value is given by the Choquet integral

\[
\int_X f \mu \left( [1/2, 3][4] \right) = \int_X f \mu \left( \frac{1}{2} + \frac{x}{3} \right) [4].
\]

In this paper, we consider the \( n \)-fuzzy measures \( \mu_1, \mu_2, \ldots, \mu_n \) and the decision operation \( \varphi : [0, 1]^n \to [0, 1] \). And in our evaluation process, the final evaluation is examined by the value

\[
\varphi \left( \left( C \right) \int_X f \mu_1, \left( C \right) \int_X f \mu_2, \ldots, \left( C \right) \int_X f \mu_n \right) \].

In the sequel, we shall only consider the case \( n = 2 \) for simplicity.

II. 2-DIMENSIONAL MEASURE AND INTEGRAL

Definition 1 2-dimensional measure \( \mu \) is the set function \( \mu : 2^X \to [0, 1] \times [0, 1] \) satisfying

\[
\mu(\emptyset) = (0, 0) \quad \text{and } \mu_X = (1, 1).
\]

Remark that the monotonicity is not assumed on \( \mu \).

Definition 2 The decision operation \( \varphi \) is a function \( \varphi : [0, 1] \times [0, 1] \to [0, 1] \) satisfying

\[
\varphi(0, 0) = 0.
\]

The score function \( f \) (of an object) is a function \( f(x) : X \to [0, 1] \). The value \( f(x_i) \) is the score (of an object) for the item \( x_i \). The integral \( I_\mu(f) = \int_X f d\mu \) is defined by the manner same to the Choquet integral as follows.

Suppose that \( f(x_1) \geq f(x_2) \geq \cdots \geq f(x_n) \geq 0 \) without loss of generality. We set \( f(x_{n+1}) = 0 \). Then \( f(x) \) is represented as

\[
f(x) = \sum_{i=1}^n (f(x_i) - f(x_{i+1})) 1_{\{x_1, x_2, \ldots, x_i\}}(x).
\]

Then the integral \( I_\mu(f) = \int_X f d\mu \) is defined by

\[
I_\mu(f) = \int_X f d\mu = \sum_{i=1}^n (f(x_i) - f(x_{i+1})) \mu_\varphi(\{x_1, x_2, \ldots, x_i\}).
\]

If we denote by \( \mu(A) = (\mu_1(A), \mu_2(A)) \), \( \mu_i : 2^X \to [0, 1] \), then

\[
I_\mu(f) = (I_{\mu_1}(f), I_{\mu_2}(f)),
\]

where

\[
I_{\mu_i}(f) = \int_X f d\mu_i = \sum_{i=1}^n (f(x_i) - f(x_{i+1})) \mu_j(\{x_1, x_2, \ldots, x_i\}),
\]

\( i = 1, 2 \).
Remark that $I_{\mu}(f) \in [0, 1]$ and $I_{\mu}(f) \in [0, 1] \times [0, 1]$. The final evaluated value (of an object) is given by $\varphi(I_{\mu}(f))$.

III. Decision operation

The decision operation $\varphi(x, y) : [0, 1] \times [0, 1] \rightarrow [0, 1]$ must be determined depending on the evaluated values $x = \int X fdy_1$ and $y = \int X fdy_2$ and also on some interactions between $x$ and $y$. We introduce the following interaction terms:

1. $s : |x - y| \ (s \in \mathbb{R})$
2. $t \sqrt{|x-y|} \ (t \geq 0)$

and we suppose that $\varphi(x, y)$ is of the form

$$\varphi(x, y) = c_1 x + c_2 y + s \cdot |x - y| + t \sqrt{|x-y|},$$

where $c_1, c_2 \geq 0$, $s \in \mathbb{R}$ and $t \geq 0$. Furthermore, to assure $\varphi(x, y) \geq 0$, we assume $c_1 \wedge c_2 \geq |s|$, and $c_1 + c_2 + t = 1$ for $\varphi(1, 1) = 1$.

Remark 3 The term $s : |x - y|$ means that, for $s < 0$, an object with $\int fdy_1 \approx \int fdy_2$ has a priority. The term $\sqrt{|x-y|}$ means that an object such that both $\int fdy_1$ and $\int fdy_2$ are different from 0 has a priority.

Proposition 4 $\varphi(x, y)$ is monotone in the following sense:

$$x_1 \leq x_2, y_1 \leq y_2 \Rightarrow \varphi(x_1, y_1) \leq \varphi(x_2, y_2).$$

Proof It is sufficient to prove the case $t = 0$. We have

$$\varphi(x_2, y_2) - \varphi(x_1, y_1)$$

$$= c_1(x_2 - x_1) + c_2(y_2 - y_1) + s \left( |x_2 - y_2| - |x_1 - y_1| \right)$$

$$\geq c_1(x_2 - x_1) + c_2(y_2 - y_1) - |s| \left( |x_2 - x_1| - |y_2 - y_1| \right)$$

$$\geq c_1(x_2 - x_1) + c_2(y_2 - y_1) - |s| (x_2 - x_1) - |s| (y_2 - y_1)$$

$$\geq 0,$$

since $c_1 \wedge c_2 \geq |s|$.

Example 5 (1) $\varphi(x, y) = x \vee y = \frac{x + y + |x - y|}{2}$

(2) $\varphi(x, y) = x \wedge y = \frac{x + y - |x - y|}{2}$

(3) $\varphi(x, y) = \frac{x + y}{2}$ (arithmetic mean)

(4) $\varphi(x, y) = \frac{\sqrt{x^2 + y^2}}{2}$ (geometric mean)

Proposition 6 If $\mu_1$ and $\mu_2$ are fuzzy measures, then $\varphi(\mu(A)) = \varphi(\mu_1(A), \mu_2(A))$ is also a fuzzy measure.

IV. Decision operation and order relation

Definition 7 A binary relation $\leq$ on $[0, 1] \times [0, 1]$ is called a comparable quasi-order if

(i) (comparability): for every $u, v \in [0, 1] \times [0, 1]$, either $u \leq v$ or $v \leq u$ holds,

(ii) (reflexivity): for every $u \in [0, 1] \times [0, 1]$, $u \leq u$ holds, and

(iii) (transitivity): $u \leq v, v \leq w$ implies $u \leq w$.

Remark that we do not suppose the antisymmetric law:

$$u \leq v \text{ and } v \leq u \text{ implies } u = v.$$

If we set $u \sim v \Leftrightarrow u \leq v$ and $v \leq u$, then $\sim$ is the equivalence relation and the quotient $[0, 1] \times [0, 1]/\sim$ becomes a totally ordered set naturally.

Using the decision operation $\varphi : [0, 1] \times [0, 1] \rightarrow [0, 1]$, we can define the order $\leq_\varphi$ in $[0, 1] \times [0, 1]$ by

$$u \leq v \Leftrightarrow \varphi(u) \leq \varphi(v) \text{ for } u, v \in [0, 1] \times [0, 1].$$

Then $\varphi : ([0, 1] \times [0, 1], \leq_\varphi) \rightarrow [0, 1]$ is monotone.

Proposition 8 The comparable quasi-order $\leq_\varphi$ is coarser than the usual order, that is,

$$u_1 \leq v_1, u_2 \leq v_2 \Rightarrow u \leq_\varphi v$$

for $u = (u_1, u_2), v = (v_1, v_2)$.

Proof The assertion follows from Proposition 4.

We say an comparable quasi-order $\leq$ on $[0, 1] \times [0, 1]$ is $\varphi$-adequate if $\leq_\varphi$ is coarser than $\leq$.

Definition 9 Let $\varphi : [0, 1] \times [0, 1] \rightarrow [0, 1]$ be a decision operation and $\mu : 2^X \rightarrow [0, 1] \times [0, 1]$ be a set function. We say $\mu$ is $\varphi$-adequate if

$$A \subset B, A, B \in 2^X \Rightarrow \mu(A) \leq_\varphi \mu(B).$$

Proposition 10 $\mu$ is $\varphi$-adequate if and only if $\varphi(\mu(A)) : 2^X \rightarrow [0, 1]$ is a fuzzy measure.

Definition 11 Let $\leq$ be any comparable quasi-order in $[0, 1] \times [0, 1]$ such that $(0, 0) \leq (1, 1)$. Then a set function $\mu : 2^X \rightarrow [0, 1] \times [0, 1]$ is called a 2-dimensional fuzzy measure on $X$ with values in $([0, 1] \times [0, 1], \leq)$ if the following conditions are satisfied:

$$\mu(\emptyset) = (0, 0), \mu(X) = (1, 1),$$

$$A \subset B, A, B \in 2^X \Rightarrow \mu(A) \leq \mu(B).$$

Proposition 12 (1) Let $\leq$ be a $\varphi$-adequate comparable quasi-order in $[0, 1] \times [0, 1]$ and let $\mu : 2^X \rightarrow ([0, 1] \times [0, 1], \leq)$ be a 2-dimensional fuzzy measure. If $\leq$ is $\varphi$-adequate, then $\varphi(\mu(A)) : 2^X \rightarrow [0, 1]$ is a fuzzy measure, that is, $\mu$ is $\varphi$-adequate.

(2) Assume that $|X| \geq 3$. Assume also that for every 2-dimensional fuzzy measure $\mu : 2^X \rightarrow ([0, 1] \times [0, 1], \leq)$, $\mu$ is $\varphi$-adequate. Then $\leq$ is $\varphi$-adequate.
Proof

(1) is clear.

(2) For every \( u, v \in [0, 1] \times [0, 1] \) with \( u \leq v \), we can find a fuzzy measure \( \mu \) such that \( \mu(x_1) = u \), \( \mu(x_1, x_2) = v \). So it follows that \( \phi(\mu(x_1)) \leq \phi(\mu(x_1, x_2)) \), that is, \( u \leq \phi(v) \).

If \( \phi(\mu(A)) \) is a fuzzy measure, we say \( \phi(\mu(A)) \) the amalgamated measure, or the distortion of \( \mu(A) \) by \( \phi(\mu) \).

Proposition 13 Let \( \mu : 2^X \rightarrow ([0, 1] \times [0, 1] \leq ) \) be a 2-dimensional fuzzy measure. Then there exists a decision operation \( \phi \) such that \( \mu \) is \( \phi \)-adequate.

Proof Let \( \mathcal{A} = \{\mu(A)|A \in 2^X\} = \{w_0 = 0 \leq w_1 \leq w_2 \leq \cdots \leq w_{2^n} = (1, 1)\} \). Then we can classify \( \mathcal{A} \) as follows:

- \( \mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \cdots \cup \mathcal{A}_L \) (disjoint sum),
- \( \mathcal{A}_k = \{w_0^k, w_1^k, \ldots, w_{2^n}^k\} \), \( w_0^k = 0, w_{2^n}^k = (1, 1) \),
- for every \( w_i^k, w_j^k \in \mathcal{A}_k \), \( w_i^k \sim w_j^k \), and
- for every \( k \leq \ell \) and for every \( u \in A_k, v \in A_{\ell} \), \( u \leq v \) holds.

We define \( \phi \) by

\[
\phi(u) = 0 \text{ if } u \in A_0 \cap A^c, \text{ and } \\
\phi(u) = \frac{k}{L} \text{ if } u \in A_k, k = 1, 2, \ldots, L.
\]

Then for \( A \subset B, A, B \subset 2^X \), we have \( \mu(A) \leq \mu(B) \). If \( \mu(A) \) and \( \mu(B) \) belongs to same class \( \mathcal{A}_k \), \( \phi(\mu(A)) = \phi(\mu(B)) \). If \( \mu(A) \leq \mu(B) \) and \( \mu(A) \sim \mu(B) \) (not equivalent), then \( \mu(A) \in A_k, \mu(B) \in A_{\ell} \) for \( k < \ell \), so that \( \phi(\mu(A)) \leq \phi(\mu(B)) \). This shows \( \mu \) is \( \phi \)-adequate.

Proposition 14 \( \phi : ([0, 1] \times [0, 1], \leq) \rightarrow \mathbb{R} \) is monotone if and only if there exists a strictly monotone function \( h : [0, 1] \times [0, 1] \rightarrow \mathbb{R} \) such that \( \phi(u) = h(\pi(u)) \), where \( \pi : [0, 1] \times [0, 1] \rightarrow [0, 1] \times [0, 1] \leq \) is the quotient mapping.

Example 15 Consider the comparable quasi-order \( \leq \) given by

\[
u \leq v \Leftrightarrow u_1 + u_2 \leq v_1 + v_2.
\]

Then \( \phi \) is monotone if and only if there exists a strictly increasing function \( h : [0, 1] \rightarrow [0, 1] \) such that

\[
\phi(s, t) = h\left(\frac{s + t}{2}\right).
\]

V. Examples

A. Evaluation by geometric mean

The evaluation process, by the geometric mean \( \sqrt[n]{r_1 r_2 \cdots r_n} \) of the score \( r_i \) for the item \( x_i \), is in our formulation as follows. Put \( \mu(A) = (\delta_{x_1}(A), \ldots, \delta_{x_n}(A)) : 2^X \rightarrow [0, 1]^n \), where \( \delta_{x_i} \) is the point mass at \( x_i \). Then we have

\[
\int f d\delta_{x_i} = f(x_i) = r_i. \text{ If we set } \varphi : [0, 1]^n \rightarrow [0, 1] \text{ by } \varphi(u_1, u_2, \ldots, u_n) = \sqrt[n]{u_1 u_2 \cdots u_n}, \text{ then we have}
\]

\[
\varphi\left(\int f d\mu\right) = \sqrt[n]{r_1 r_2 \cdots r_n}.
\]

B. Evaluation taking account of the dual measure \( \mu_d \)

Let \( \mu_d \) be the dual measure of \( \mu, \mu_d(A) = 1 - \mu(A^c) \). Set \( \mu(A) = (\mu(A), \mu_d(A)) \). Then the value \( \varphi(\int f d\mu) \) reflects a contribution of \( \mu_d \).

The meaning of \( (C) \int_X f d\mu_d \) is as follows. The score function \( f(x) \) can be considered as a degree of satisfaction, so \( 1 - f(x) \) be a degree of dissatisfaction. Consequently, the Choquet integral \( (C) \int_X (1 - f)d\mu_d \) is a total score of dissatisfaction. It is desirable to minimize \( (C) \int_X (1 - f)d\mu_d \), equivalently, to maximize \( 1 - (C) \int_X (1 - f)d\mu_d \) to reduce the dissatisfaction.

REFERENCES