Subjective Evaluation Process by Multi-Dimensional Fuzzy Measure

Aoi Honda, Junji Yamagami and Yoshiaki Okazaki
Kyushu Institute of Technology

Department of Systems Innovation and Informatics,
Faculty of Computer Science and Systems Engineering
Kyushu Institute of Technology, Kawazu, Iizuka 820-8502, JAPAN
email: aoi@ces.kyutech.ac.jp

Abstract— **We investigate the structure of an evaluation process using a multi-dimensional fuzzy measure formulated as follows. Let $\mu_1, \mu_2, \dots, \mu_n$ be the measures of subjective evaluation and $f(x_i)$ be a score function (of an object in consideration) for items $x_i, i = 1, 2, \dots, k$. We consider the evaluation process such that the final evaluative value (of an object) is given by $\varphi\left((C)\int fd\mu_1, \dots, (C)\int fd\mu_n\right)$, where $(C)\int fd\mu_i$ is the Choquet integral and $\varphi : [0,1]^n \to [0,1]$ be a decision operation.**

As an example, we give an evaluation process taking account of the dual measure μ_d, the 2-dimensional fuzzy measure $\boldsymbol{\mu} = (\mu, \mu_d)$, reducing the degree of dissatisfaction.

Keywords: distorted probability, subjective evaluation, multi-dimensional fuzzy measure

I. Introduction

The measure of subjective evaluation is formulated as a fuzzy measure on the set of items $X = \{x_1, x_2, \dots, x_k\}$. A fuzzy measure is a set function $\mu : 2^X \to [0,1]$ satisfying

$$\mu(\emptyset) = 0, \mu(X) = 1, \text{ and}$$
$$\text{if } A \subset B, A, B, \in 2^X, \text{ then } \mu(A) \le \mu(B) \quad \text{(monotonicity)}.$$

If an object has a score function $f(x) : X \to [0,1]$, then its evaluated value is given by the Choquet integral $(C)\int_X fd\mu$ ([1][2][3][4]).

In this paper, we consider the n-fuzzy measures $\mu_1, \mu_2, \cdots, \mu_n$ and the decision operation $\varphi : [0,1]^n \to [0,1]$. And in our evaluation process, the final evaluation is examined by the value $\varphi\left((C)\int_X fd\mu_1, (C)\int_X fd\mu_2, \dots, (C)\int_X fd\mu_n\right)$. In the sequel, we shall only consider the case $n = 2$ for simplicity.

II. 2-dimensional measure and integral

Definition 1 2-dimensional measure $\boldsymbol{\mu}$ is the set function $\boldsymbol{\mu} : 2^X \to [0,1] \times [0,1]$ satisfying

$$\boldsymbol{\mu}(\emptyset) = (0,0) \text{ and } \boldsymbol{\mu}(X) = (1,1).$$

Remark that the monotonicity is not assumed on $\boldsymbol{\mu}$.

Definition 2 The decision operation φ is a function $\varphi : [0,1] \times [0,1] \to [0,1]$ satisfying

$$\varphi(0,0) = 0.$$

The score function f (of an object) is a function $f(x) : X \to [0,1]$. The value $f(x_i)$ is the score (of an object) for the item x_i. The integral $I_{\boldsymbol{\mu}}(f) = \int_X fd\boldsymbol{\mu}$ is defined by the manner same to the Choquet integral as follows.

Suppose that $f(x_1) \ge f(x_2) \ge \cdots \ge f(x_n) \ge 0$ without loss of generality. We set $f(x_{n+1}) = 0$. Then $f(x)$ is represented as

$$f(x) = \sum_{i=1}^{n} \left(f(x_i) - f(x_{i+1})\right) \mathbb{1}_{\{x_1, x_2, \dots, x_i\}}(x).$$

Then the integral $I_{\boldsymbol{\mu}}(f) = \int_X fd\boldsymbol{\mu}$ is defined by

$$\begin{aligned} I_{\boldsymbol{\mu}}(f) &= \int_X fd\boldsymbol{\mu} \\ &= \sum_{i=1}^{n} \left(f(x_i) - f(x_{i+1})\right) \boldsymbol{\mu}\left(\{x_1 x_2 \cdots x_i\}\right). \end{aligned}$$

If we denote by $\boldsymbol{\mu}(A) = (\mu_1(A), \mu_2(A))$, $\mu_i : 2^X \to [0,1]$, then

$$I_{\boldsymbol{\mu}}(f) = \left(I_{\mu_1}(f), I_{\mu_2}(f)\right),$$

where

$$\begin{aligned} I_{\mu_j}(f) &= \int_X fd\mu_j \\ &= \sum_{i=1}^{n} \left(f(x_i) - f(x_{i+1})\right) \mu_j\left(\{x_1 x_2 \cdots x_i\}\right), \end{aligned}$$

$i = 1, 2.$

Remark that $I_{\mu_j}(f) \in [0,1]$ and $I_{\boldsymbol{\mu}}(f) \in [0,1]\times[0,1]$. The final evaluated value (of an object) is given by $\varphi\left(I_{\boldsymbol{\mu}}(f)\right)$.

III. Decision operation

The decision operation $\varphi(x,y) : [0,1]\times[0,1] \to [0,1]$ must be determined depending on the evaluated values $x = \int_X f d\mu_1$ and $y = \int_X f d\mu_2$ and also on some interactions between x and y. We introduce the following interaction terms;

(i) $s \cdot |x-y| \quad (s \in I\!R)$
(ii) $t\sqrt{xy} \qquad (t \geq 0)$,

and we suppose that $\varphi(x,y)$ is of the form

$$\varphi(x,y) = c_1 x + c_2 y + s\cdot|x-y| + t\sqrt{xy},$$

where $c_1, c_2 \geq 0$, $s \in I\!R$ and $t \geq 0$. Furthermore, to assure $\varphi(x,y) \geq 0$, we assume $c_1 \wedge c_2 \geq |s|$, and $c_1 + c_2 + t = 1$ for $\varphi(1,1) = 1$.

Remark 3 The term $s\cdot|x-y|$ means that, for $s<0$, an object with $\int f d\mu_1 \approx \int f d\mu_2$ has a priority. The term $\sqrt{xy}$ means that an object such that both $\int f d\mu_1$ and $\int f d\mu_2$ are different from 0 has a priority.

Proposition 4 $\varphi(x,y)$ is monotone in the following sense:

$$x_1 \leq x_2, y_1 \leq y_2 \Rightarrow \varphi(x_1,y_1) \leq \varphi(x_2,y_2).$$

Proof It is sufficient to prove the case $t=0$. We have

$$\begin{aligned}
&\varphi(x_2,y_2) - \varphi(x_1,y_1)\\
&\quad = c_1(x_2-x_1) + c_2(y_2-y_1) + s\left(|x_2-y_2| - |x_1-y_1|\right)\\
&\quad \geq c_1(x_2-x_1) + c_2(y_2-y_1) - |s|\,|(x_2-x_1)-(y_2-y_1)|\\
&\quad \geq c_1(x_2-x_1) + c_2(y_2-y_1) - |s|(x_2-x_1) - |s|(y_2-y_1)\\
&\quad \geq 0,
\end{aligned}$$

since $c_1 \wedge c_2 \geq |s|$.

Example 5 (1) $\varphi(x,y) = x \vee y = \dfrac{x+y+|x-y|}{2}$

(2) $\varphi(x,y) = x \wedge y = \dfrac{x+y-|x-y|}{2}$

(3) $\varphi(x,y) = \dfrac{x+y}{2}$ (arithmetic mean)

(4) $\varphi(x,y) = \sqrt{xy}$ (geometric mean)

Proposition 6 If μ_1 and μ_2 are fuzzy measures, then $\varphi(\boldsymbol{\mu}(A)) = \varphi(\mu_1(A), \varphi_2(A))$ is also a fuzzy measure.

IV. Decision operation and order relation

Definition 7 A binary relation $\boldsymbol{u} \leq \boldsymbol{v}$ on $[0,1]\times[0,1]$ is called a comparable quasi-order if

(i) (comparability): for every $\boldsymbol{u}, \boldsymbol{v} \in [0,1]\times[0,1]$, either $\boldsymbol{u} \leq \boldsymbol{v}$ or $\boldsymbol{v} \leq \boldsymbol{u}$ holds,
(ii) (reflexivity): for every $\boldsymbol{u} \in [0,1]\times[0,1]$, $\boldsymbol{u} \leq \boldsymbol{u}$ holds, and
(iii) (transitivity): $\boldsymbol{u} \leq \boldsymbol{v}, \boldsymbol{v} \leq \boldsymbol{w}$ implies $\boldsymbol{u} \leq \boldsymbol{w}$.

Remark that we do not suppose the antisymmetric law:

$$\boldsymbol{u} \leq \boldsymbol{v} \text{ and } \boldsymbol{v} \leq \boldsymbol{v} \text{ implies } \boldsymbol{u} = \boldsymbol{v}.$$

If we set $\boldsymbol{u} \sim \boldsymbol{v} \Leftrightarrow \boldsymbol{u} \leq \boldsymbol{v}$ and $\boldsymbol{v} \leq \boldsymbol{u}$, then $\sim$ is the equivalence relation and the quotient $[0,1]\times[0,1]/\!\sim$ becomes a totally ordered set naturally.

Using the decision operation $\varphi : [0,1]\times[0,1] \to [0,1]$, we can define the order $\leq_\varphi$ in $[0,1]\times[0,1]$ by

$$\boldsymbol{u} \leq_\varphi \boldsymbol{v} \Leftrightarrow \varphi(\boldsymbol{u}) \leq \varphi(\boldsymbol{v}) \text{ for } \boldsymbol{u}, \boldsymbol{v} \in [0,1]\times[0,1].$$

Then $\varphi : ([0,1]\times[0,1], \leq_\varphi) \to [0,1]$ is monotone.

Proposition 8 The comparable quasi-order $\leq_\varphi$ is coarser than the usual order, that is,

$$u_1 \leq v_1, u_2 \leq v_2 \Rightarrow \boldsymbol{u} \leq_\varphi \boldsymbol{v}$$

for $\boldsymbol{u} = (u_1, u_2), \boldsymbol{v} = (v_1, v_2)$.

Proof The assertion follows from Proposition 4.

We say an comparable quasi-order $\leq$ on $[0,1]\times[0,1]$ is φ-adequate if $\leq_\varphi$ is coarser than $\leq$.

Definition 9 Let $\varphi : [0,1]\times[0,1] \to [0,1]$ be a decision operation and $\boldsymbol{\mu} : 2^X \to [0,1]\times[0,1]$ be a set function. We say $\boldsymbol{\mu}$ is φ-adequate if

$$A \subset B, A, B \in 2^X \Rightarrow \boldsymbol{\mu}(A) \leq_\varphi \boldsymbol{\mu}(B).$$

Proposition 10 $\boldsymbol{\mu}$ is φ-adequate if and only if $\varphi(\boldsymbol{\mu}(A)) : 2^X \to [0,1]$ is a fuzzy measure.

Definition 11 Let $\leq$ be any comparable quasi-order in $[0,1]\times[0,1]$ such that $(0,0) \leq (1,1)$. Then a set function $\boldsymbol{\mu} : 2^X \to [0,1]\times[0,1]$ is called a 2-dimensional fuzzy measure on X with values in $([0,1]\times[0,1], \leq)$ if the following conditions are satisfied:

$$\begin{aligned}
&\boldsymbol{\mu}(\emptyset) = (0,0),\ \boldsymbol{\mu}(X) = (1,1), \quad \text{and}\\
&A \subset B, A, B \in 2^X \Rightarrow \boldsymbol{\mu}(A) \leq \boldsymbol{\mu}(B).
\end{aligned}$$

Proposition 12 (1) Let $\leq$ be a φ-adequate comparable quasi-order in $[0,1]\times[0,1]$ and let $\boldsymbol{\mu} : 2^X \to ([0,1]\times[0,1], \leq)$ be a 2-dimensional fuzzy measure. If $\leq$ is φ-adequate, then $\varphi(\boldsymbol{\mu}(A)) : 2^X \to [0,1]$ is a fuzzy measure, that is, $\boldsymbol{\mu}$ is φ-adequate.

(2) Assume that $|X| \geqq 3$. Assume also that for every 2-dimensional fuzzy measure $\boldsymbol{\mu} : 2^X \to ([0,1]\times[0,1], \leq)$, $\boldsymbol{\mu}$ is φ-adequate. Then $\leq$ is φ-adequate.

Proof

(1) is clear.

(2) For every $\boldsymbol{u}, \boldsymbol{v} \in [0,1] \times [0,1]$ with $\boldsymbol{u} \leq \boldsymbol{v}$, we can find a fuzzy measure $\boldsymbol{\mu}$ such that $\boldsymbol{\mu}(\{x_1\}) = \boldsymbol{u}$, $\boldsymbol{\mu}(\{x_1, x_2\}) = \boldsymbol{v}$. So it follows that $\varphi(\boldsymbol{\mu}(\{x_1\}) \leqq \varphi(\boldsymbol{\mu}(\{x_1, x_2\})$, that is, $\boldsymbol{u} \leq_\varphi \boldsymbol{v}$.

If $\varphi(\boldsymbol{\mu}(A))$ is a fuzzy measure, we say $\varphi(\boldsymbol{\mu}(A))$ the amalgamated measure, or the distortion of $\boldsymbol{\mu}(A)$ by φ ([5]).

Proposition 13 Left $\boldsymbol{\mu} : 2^X \to ([0,1] \times [0,1], \leq)$ be a 2-dimensional fuzzy measure. Then there exists a decision operation φ such that $\boldsymbol{\mu}$ is φ-adequate.

Proof Let $\mathcal{A} = \{\boldsymbol{\mu}(A) | A \in 2^X\} = \{\boldsymbol{w}_0 = \boldsymbol{0} \leq \boldsymbol{w}_1 \leq \boldsymbol{w}_2 \leq \cdots \leq \boldsymbol{w}_{2^n} = (1,1)\}$. Then we can classify $\mathcal{A}$ as follows:

- $\mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \cup \cdots \mathcal{A}_L$ (disjoint sum),
- $\mathcal{A}_k = \{\boldsymbol{w}_1^k, \boldsymbol{w}_2^k, \ldots, \boldsymbol{w}_{p_k}^k\}, \boldsymbol{w}_1^0 = \boldsymbol{0}, \boldsymbol{w}_{p_L}^L = (1,1)$,
- for every $\boldsymbol{w}_i^k, \boldsymbol{w}_j^k \in \mathcal{A}_k$, $\boldsymbol{w}_i^k \sim \boldsymbol{w}_j^k$, and
- for every $k < \ell$ and for every $\boldsymbol{u} \in \mathcal{A}_k, \boldsymbol{v} \in \mathcal{A}_\ell, \boldsymbol{u} \leq \boldsymbol{v}$ holds.

We define φ by

$$\varphi(\boldsymbol{u}) = 0 \text{ if } \boldsymbol{u} \in \mathcal{A}_0 \cap \mathcal{A}^c, \quad \text{and}$$

$$\varphi(\boldsymbol{u}) = \frac{k}{L} \quad \text{if} \quad \boldsymbol{u} \in \mathcal{A}_k, k = 1, 2, \ldots, L.$$

Then for $A \subset B, A, B, \in 2^X$, we have $\boldsymbol{\mu}(A) \leq \boldsymbol{\mu}(B)$. If $\boldsymbol{\mu}(A)$ and $\boldsymbol{\mu}(B)$ belongs to same class $\mathcal{A}_k$, $\varphi(\boldsymbol{\mu}(A)) = \varphi(\boldsymbol{\mu}(B))$. If $\boldsymbol{\mu}(A) \leq \boldsymbol{\mu}(B)$ and $\boldsymbol{\mu}(A) \nsim \boldsymbol{\mu}(B)$ (not equivalent), then $\boldsymbol{\mu}(A) \in \mathcal{A}_k$, $\boldsymbol{\mu}(B) \in \mathcal{A}_\ell$ for $k < \ell$, so that $\varphi(\boldsymbol{\mu}(A)) \leq \varphi(\boldsymbol{\mu}(B))$. This shows $\boldsymbol{\mu}$ is φ-adequate.

Proposition 14 $\varphi : ([0,1] \times [0,1], \leq) \to \mathbb{R}$ is monotone if and only if there exits a strictly monotone function $h : [0,1 \times [0,1]/\sim \to \mathbb{R}$ such that $\varphi(\boldsymbol{u}) = h(\pi(\boldsymbol{u}))$, where $\pi : [0,1] \times [0,1] \to [0,1] \times [0,1]/\sim$ is the quotient mapping.

Example 15 Consider the comparable quasi-order $\leq$ given by

$$\boldsymbol{u} \leq \boldsymbol{v} \Leftrightarrow u_1 + u_2 \leq v_1 + v_2.$$

Then φ is monotone if and only if there exists a strictly increasing function $h : [0,1] \to [0,1]$ such that

$$\varphi(s,t) = h\left(\frac{s+t}{2}\right).$$

V. Examples

A. Evaluation by geometric mean

The evaluation process, by the geometric mean $\sqrt[n]{r_1 r_2 \cdots r_n}$ of the score r_i for the item x_i, is in our formulation as follows. Put $\boldsymbol{\mu}(A) = (\delta_{x_1}(A), \ldots, \delta_{x_n}(A)) : 2^X \to [0,1]^n$, where δ_{x_i} is the point mass at x_i. Then we have $\int f d\delta_{x_i} = f(x_i) = r_i$. If we set $\varphi : [0,1]^n \to [0,1]$ by $\varphi(u_1, u_2, \ldots, u_n) = \sqrt[n]{u_1 u_2 \cdots u_n}$ then we have $\varphi\left(\int f d\boldsymbol{\mu}\right) = \sqrt[n]{r_1 r_2 \cdots r_n}$.

B. Evaluation taking account of the dual measure μ_d

Let μ_d be the dual measure of $\mu, \mu_d(A) = 1 - \mu(A^c)$. Set $\boldsymbol{\mu}(A) = (\mu(A), \mu_d(A))$. Then the value $\varphi\left(\int f d\boldsymbol{\mu}\right)$ reflects a contribution of μ_d.

The meaning of $(C)\int_X f d\mu_d$ is as follows. The score function $f(x)$ can be considered as a degree of satisfaction, so $1 - f(x)$ be a degree of dissatisfaction. Consequently, the Choquet integral $(C)\int_X (1-f) d\mu$ is a total score of dissatisfaction. It is desirable to minimize $(C)\int_X (1-f) d\mu$, equivalently, to maximize $1 - (C)\int_X (1-f) d\mu = (C)\int_X f d\mu_d$ to reduce the dissatisfaction.

References

[1] A. Honda, T, Nakano and Y. Okazaki, Subjective evaluation based on distorted probability, Joint Int. Conf. on Soft Computing and Intelligent Systems and 3d Int. Symp. on Advanced Intelligent Systems, Tsukuba, Japan, October 2002.

[2] Endre Pap, *Null-Additive Set Function*, Kluwer Academic Publishers, 1995, Dordrech-Boston-London.

[3] M. Sugeno and T. Murofushi, *Fuzzy Measure*, Nikkan-kogyo-Shinbunsha, 1993, (in Japanese).

[4] Wang and G. J. Klir, *Fuzzy Measure Theory*, Plenum Press, 1992, New York and London.

[5] J. Yamagami and A. Honda, Fuzzy Measures of Two Dimensions and Distortion, 8-th workshop of Heart and Mind 2003, pp. 76–80, 2003.