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ABSOLUTE CONTINUITY OF DISCRETE ONE-SIDED RANDOM TRANSLATION

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ABSTRACT. Let $\mathbf{X}$ be an IID random sequence and $\mathbf{Y}$ be an independent random sequence which is also independent of $\mathbf{X}$, and $\mu_{\mathbf{X}}$ and $\mu_{\mathbf{X}+\mathbf{Y}}$ are the probability measures on the sequence space induced by $\mathbf{X}$ and $\mathbf{X}+\mathbf{Y}$, respectively. The aim of this paper is to characterize $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$(mutually absolutely continuous) in terms of $\mu_{\mathbf{Y}}$ in the case where $\mathbf{X}$ and $\mathbf{Y}$ take values in non-negative integers. First, we give necessary or sufficient conditions for general $\mathbf{X}$ and $\mathbf{Y}$ by introducing a ratio function. Second, we give a necessary and sufficient condition for $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ when $\mathbf{X}$ obeys a geometric distribution, and give several examples.

1 Introduction

Throughout this paper $\mathbf{X} = \{X_k\}_{k=1}^{\infty}$ denotes an independent identically distributed(IID) random sequence defined on a probability space $(\Omega, \mathcal{F}, I\!P)$, and $\mathbf{Y} = \{Y_k\}_{k=1}^{\infty}$ an independent random sequence on $(\Omega, \mathcal{F}, I\!P)$ which is also independent of $\mathbf{X}$. Denote by $\mu_{\mathbf{X}}$ and $\mu_{\mathbf{X}+\mathbf{Y}}$ the probability measures on the sequence space induced by $\mathbf{X}$ and $\mathbf{X}+\mathbf{Y} = \{X_k + Y_k\}_{k=1}^{\infty}$, respectively. Furthermore, assume $\mu_{X_k+Y_k} \sim \mu_{X_k}$(mutually absolutely continuous) for each pair of marginal distributions $\mu_{X_k+Y_k}$ and $\mu_{X_k}, k \geq 1$. Then Kakutani's dichotomy theorem implies either $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ or $\mu_{\mathbf{X}+\mathbf{Y}} \perp \mu_{\mathbf{X}}$(singular). The problem is to characterize $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ in terms of $\mu_{\mathbf{Y}}$.

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In [2], they discussed the above problem when $\mathbf{X}$ and $\mathbf{Y}$ take values in non-negative real numbers, that is, $X_k, Y_k \in I\!R_+ = [0,\infty)$. They gave a necessary and sufficient condition for $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ when $\mathbf{X}$ obeys an exponential distribution and $\mathbf{Y}$ is arbitrary. Furthermore, they discussed the 0-1 valued Y_k and gave a necessary or sufficient conditions for $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$.

In this paper, we shall discuss the case where $\mathbf{X}$ and $\mathbf{Y}$ take values in non-negative integers, that is, $X_k, Y_k \in \mathbf{N}_0 = \{0,1,2,\dots\}$. We denote the density functions of X_1 and Y_k by

$$f(n) = I\!P(X_1 = n) \quad \text{and} \quad g_k(m) = I\!P(Y_k = m), \ n, m \in \mathbf{N}_0,$$

respectively. We assume $f(n) > 0$ for every $n \in \mathbf{N}_0$ and $g_k(0) > 0$ for every $k \geq 1$ and define $f(n) = 0$ if $n < 0$ for convenience.

Sato and Tamashiro[5] introduced the informations of $\mathbf{X}$ defined by:

$$J(\mathbf{X};\ell) := \sum_{n=0}^{\infty} \frac{(f(n-\ell) - f(n))^2}{f(n)} < +\infty, \ \ell \geq 1,$$

which we shall call the discrete Fisher information(Shepp[6]) of order ℓ.

In Section 2, We shall introduce a ratio function $\rho(n)$ of $\mathbf{X}$ and discuss the relations among $\rho(n)$, $Z_k(n)$ and $J(\mathbf{X};\ell)$, where $Z_k(n)$ is defined by

$$\begin{aligned} Z_k(n) &:= \frac{d\mu_{X_k+Y_k}}{d\mu_{X_k}}(n) - 1 \\ &= \frac{\sum_{m=0}^{\infty} f(n-m) g_k(m)}{f(n)} - 1 \\ &= \frac{\sum_{m=1}^{\infty} \left(f(n-m) - f(n)\right) g_k(m)}{f(n)}. \end{aligned}$$

We shall show that if $\rho(n)$ is non-increasing, then $f(n)$ is log-concave and $Z_k(n)$ is non-decreasing(Lemma 5).

If $Z_k(n)$ is non-decreasing, then we define

$$\gamma_k := \begin{cases} \infty, & \text{if } \sup_n Z_k(n) \leq 1, \\ \min\{n \in \mathbf{N}_0 \mid Z_k(n) > 1\}, & \text{otherwise,} \end{cases}$$

and prove the following theorem.

Theorem 1. Assume that $\mathbf{X}$ and $\mathbf{Y}$ take values in $\mathbf{N}_0$, and $Z_k(n)$ is non-decreasing. Then $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ implies

$$\sum_k \mathbb{P}(\gamma_k \le Y_k) < \infty.$$

Furthermore, we shall give a sufficient condition for $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ in terms of the discrete Fisher informations as follows.

Theorem 2. Assume that $\mathbf{X}$ and $\mathbf{Y}$ take values in $\mathbf{N}_0$. If there exists a sequence of positive numbers $\{L_k\}_{k=1}^{\infty}$ such that

$$\sum_{k=1}^{\infty} \mathbb{P}(Y_k \ge L_k) < \infty \tag{1}$$

and

$$\sum_{k=1}^{\infty} \sup_{0<\ell<L_k} J(\mathbf{X};\ell)\ \mathbb{P}(0 < Y_k < L_k)^2 < \infty, \tag{2}$$

then $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$.

This is a generalizaion of Sato and Tamashiro[5]'s Theorem 2.1.

In Section 4, we shall specify the distribution of X_k to a geometric distribution and give a necessary and sufficient condition for $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ by similar discussions as in [2].

Theorem 3. Let $\mathbf{X}$ be an $\mathbf{N}_0$-valued IID random sequence and assume that $\mathbf{X}$ obeys a geometric distribution, that is, $f(n) = pq^n,\ n \in \mathbf{N}_0$ for $p, q > 0$ $(p + q = 1)$ and let $Y_k, k \ge 1$, take values in $\mathbf{N}_0$. Then $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ if and only if the following three series converge:

$$\sum_{k=1}^{\infty} \mathbb{P}(\gamma_k \le Y_k) < \infty, \tag{3}$$

$$\sum_{k=1}^{\infty} q^{\gamma_k} < \infty \tag{4}$$

and

$$\sum_{k=1}^{\infty} \sum_{m=0}^{\gamma_k - 1} q^{-m} \mathbb{P}(m < Y_k < \gamma_k)^2 < \infty, \tag{5}$$

where

$$\gamma_k = \begin{cases} \infty, & \text{if } \mathbb{E}[q^{-Y_k}] \leq 2, \\ \min\left\{n \in \mathbf{N}_0 \mid \mathbb{E}[q^{-Y_k} : Y_k \leq n] > 2\right\}, & \text{otherwise} \end{cases} \tag{6}$$

We shall also give several corollaries of the above theorem.

2 Ratio function $\rho(n)$

In this section, we shall discuss the relation among the ratio function $\rho(n)$, $Z_k(n)$ and the discrete Fisher informations. We define $\rho(n)$ by

$$\rho(n) := \frac{f(n)}{f(n-1)}, \ n \geq 1.$$

Example 4. (i) When $\mathbf{X}$ obeys a geometric distribution, we have $\rho(n) = q$.
(ii) When $\mathbf{X}$ obeys a Poisson distribution with mean $\lambda > 0$, we have $\rho(n) = \dfrac{\lambda}{n}$.

Lemma 5. If $\rho(n)$ is non-increasing, then $f(n)$ is log-concave and $Z_k(n)$ is non-decreasing.

Proof. Assume $\rho(n)$ is non-increasing. Then we have

$$0 \leq \rho(n) - \rho(n+1) = \frac{f(n)}{f(n-1)} - \frac{f(n+1)}{f(n)},$$

that is,

$$\frac{f(n-1)}{f(n)} \leq \frac{f(n)}{f(n+1)},$$

which implies the log-concavity of $f(n)$.

On the other hand, we have

$$\begin{aligned} & Z_k(n) - Z_k(n+1) \\ = & \left(\sum_{\ell=0}^{\infty} \frac{f(n-\ell)}{f(n)} g_k(\ell) - 1\right) - \left(\sum_{\ell=0}^{\infty} \frac{f(n+1-\ell)}{f(n+1)} g_k(\ell) - 1\right) \\ = & \sum_{\ell=1}^{\infty} \left(\frac{f(n-\ell)}{f(n)} - \frac{f(n+1-\ell)}{f(n+1)}\right) g_k(\ell). \end{aligned}$$

Noting

$$\frac{f(n-\ell)}{f(n)} = \frac{f(n-1)}{f(n)}\frac{f(n-2)}{f(n-1)}\cdots\frac{f(n-\ell)}{f(n-\ell+1)}$$

and

$$\frac{f(n+1-\ell)}{f(n+1)} = \frac{f(n)}{f(n+1)}\frac{f(n-1)}{f(n)}\cdots\frac{f(n+1-\ell)}{f(n+2-\ell)},$$

we have for any ℓ, $\frac{f(n-\ell)}{f(n)} \leq \frac{f(n+1-\ell)}{f(n+1)}$, hence $Z_k(n)-Z_k(n+1) \leq 0$.

The discrete Fisher informations and $\rho(n)$ are related as follows.

Lemma 6. If there exists $\varepsilon > 0$ such that $\rho(n) \geq \varepsilon$ for any n, then we have $J(\mathbf{X};\ell) < +\infty$ for every $\ell = 1, 2, \ldots$.

Proof. Since $\rho(n) \geq \varepsilon$ for any n, we have $\sup_n \frac{1}{\rho(n)^2} \leq \frac{1}{\varepsilon^2} < \infty$ and for any ℓ

$$\begin{aligned}\frac{f(n-\ell)^2}{f(n)} &= \frac{f(n-\ell)^2}{f(n-\ell+1)^2}\frac{f(n-\ell+1)^2}{f(n-\ell+2)^2}\cdots\frac{f(n-1)^2}{f(n)^2}f(n)\\ &= \frac{1}{\rho(n-\ell+1)^2}\frac{1}{\rho(n-\ell+2)^2}\cdots\frac{1}{\rho(n)^2}f(n).\end{aligned}$$

Therefore we have

$$\begin{aligned}J(\mathbf{X};\ell) &= \sum_{n=0}^{\infty}\frac{(f(n-\ell)-f(n))^2}{f(n)} = \sum_{n=0}^{\infty}\frac{f(n-\ell)^2}{f(n)} - 1\\ &\leq \frac{1}{\varepsilon^{2\ell}}\sum_{n=0}^{\infty}f(n) = \frac{1}{\varepsilon^{2\ell}} < \infty.\end{aligned}$$

The converse of Lemma 6 does not hold. For instance, if $\mathbf{X}$ obeys a Poisson distribution, then, although $J(\mathbf{X};\ell) < \infty$ for any ℓ, we have $\inf_n \rho(n) = 0$.

Next, we shall give an example such that $J(\mathbf{X};\ell) = \infty$.

Example 7. Define

$$f(n) = \begin{cases} C^{-1}a^m, & \text{if } n = 2m, \\ C^{-1}b^m, & \text{if } n = 2m+1, \end{cases}$$

where $0 < a, b < 1, m \in \mathbf{N}_0$, $\dfrac{a^2}{b} \geq 1$ and $C = \dfrac{1}{1-a} + \dfrac{1}{1-b}$. Then we have $\sup_n \rho(n) = \infty$, $\inf_n \rho(n) = 0$ and

$$\begin{cases} J(\mathbf{X};\ell) = \infty, & \ell \text{ is odd,} \\ J(\mathbf{X}\ \ell) < \infty, & \ell \text{ is even.} \end{cases}$$

In fact, in the case where ℓ is an odd number, we have

$$\begin{aligned} J(\mathbf{X};\ell) &= \sum_{n=0}^{\infty} \frac{f(n-\ell)^2}{f(n)} - 1 \geq \sum_{n \text{ is odd}} \frac{f(n-\ell)^2}{f(n)} - 1 \\ &= \sum_{m=0}^{\infty} \frac{C^{-2}a^{2m-\ell+1}}{C^{-1}b^m} - 1 \\ &= C^{-1}a^{1-\ell} \sum_{m=0}^{\infty} \left(\frac{a^2}{b}\right)^m - 1 = \infty, \end{aligned}$$

and in the case where ℓ is an even number, we have

$$\begin{aligned} J(\mathbf{X};\ell) &= \sum_{n=0}^{\infty} \frac{f(n-\ell)^2}{f(n)} - 1 \\ &= \sum_{n \text{ is odd}} \frac{f(n-\ell)^2}{f(n)} + \sum_{n \text{ is even}} \frac{f(n-\ell)^2}{f(n)} - 1 \\ &= \sum_{m=0}^{\infty} \frac{C^{-2}b^{2m-\ell}}{C^{-1}b^m} + \sum_{m=0}^{\infty} \frac{C^{-2}a^{2m-\ell}}{C^{-1}a^m} - 1 \\ &= \frac{C^{-1}}{b^\ell} \sum_{m=0}^{\infty} b^m + \frac{C^{-1}}{a^\ell} \sum_{m=0}^{\infty} a^m - 1 < \infty. \end{aligned}$$

3 Random translation

To prove Theorem 1, we use Kitada and Sato's criterion for $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$.

Lemma 8. (Kitada and Sato[4], Theorem 2) $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ if and only if

$$\text{(KS.1)} \qquad \sum_{k=1}^{\infty} \mathbb{E}[Z_k(X_k); Z_k(X_k) > 1] < \infty$$

and

$$\text{(KS.2)} \qquad \sum_{k=1}^{\infty} \mathbb{E}[Z_k(X_k)^2; Z_k(X_k) \le 1] < \infty.$$

Proof of Theorem 1. Assume $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$. Then by Lemma 8, we have (KS.1), so that

$$\begin{aligned}
\infty &> \sum_{k=1}^{\infty} \sum_{n\in\{n|Z_k(n)>1\}} \mathbb{E}[f(n-Y_k)-f(n)] \\
&= \sum_{k=1}^{\infty}\sum_{n=\gamma_k}^{\infty}\sum_{\ell=0}^{\infty}(f(n-\ell)-f(n))\,g_k(\ell) \\
&= \sum_{k=1}^{\infty}\left(\sum_{n=\gamma_k}^{\infty}\sum_{\ell=0}^{\infty} f(n-\ell)g_k(\ell) - \sum_{n=\gamma_k}^{\infty}\sum_{\ell=0}^{\infty} f(n)g_k(\ell)\right).
\end{aligned}$$

Since

$$\begin{aligned}
&\sum_{n=\gamma_k}^{\infty}\sum_{\ell=1}^{\infty} f(n-\ell)g_k(\ell) \\
&= \sum_{n=\gamma_k}^{\infty}\sum_{\ell=0}^{n} f(n-\ell)g_k(\ell) = \sum_{n=\gamma_k}^{\infty}\sum_{\ell=0}^{n} f(\ell)g_k(n-\ell) \\
&= \sum_{n=\gamma_k}^{\infty}\sum_{\ell=0}^{\gamma_k-1} f(\ell)g_k(n-\ell) + \sum_{n=\gamma_k}^{\infty}\sum_{\ell=\gamma_k}^{n} f(\ell)g_k(n-\ell),
\end{aligned}$$

we have

$$\begin{aligned}
\infty &> \sum_{k=1}^{\infty} \mathbb{E}[Z_k(X_k) : Z_k(X_k) > 1] \\
&= \sum_{k}\left(\sum_{n=0}^{\gamma_k-1}\sum_{\ell=\gamma_k}^{\infty} f(n)g_k(\ell-n)\right. \\
&\qquad \left. + \sum_{n=\gamma_k}^{\infty}\sum_{\ell=1}^{\infty} f(n)g_k(\ell) - \sum_{n=\gamma_k}^{\infty}\sum_{\ell=1}^{\infty} f(n)g_k(\ell)\right) \\
&= \sum_{k=1}^{\infty}\sum_{n=0}^{\gamma_k-1}\sum_{\ell=\gamma_k}^{\infty} f(n)g_k(\ell-n) \ge \sum_{k=1}^{\infty}\sum_{\ell=1}^{\infty} f(0)g_k(\ell)
\end{aligned}$$

$$= \quad f(0)\sum_{k=1}^{\infty} I\!P(Y_k \geq \gamma_k)$$

This completes the proof.

Proof of Theorem 2. By Kakutani[3]'s criterion, we have $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ if and only if

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\left|\sqrt{f(n)} - \sqrt{I\!E\left[f(n-Y_k)\right]}\right|^2 < \infty. \tag{7}$$

Since

$$\left|\sqrt{f(n)} - \sqrt{I\!E\left[f(n-Y_k)\right]}\right|^2 = \frac{\left(f(n) - I\!E\left[f(n-Y_k)\right]\right)^2}{\left(\sqrt{f(n)} + \sqrt{I\!E\left[f(n-Y_k)\right]}\right)^2}$$

and for any $a, b \geq 0$, $a + b \leq \left(\sqrt{a} + \sqrt{b}\right)^2 \leq 2(a+b)$, (7) is equivalent to

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{\left(f(n) - I\!E\left[f(n-Y_k)\right]\right)^2}{f(n) + I\!E\left[f(n-Y_k)\right]} < \infty.$$

We have

$$\begin{aligned}
&\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\tfrac{(f(n)-I\!E[f(n-Y_k)])^2}{f(n)+I\!E[f(n-Y_k)]}\\
&\quad= \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\tfrac{(I\!E[f(n)-f(n-Y_k)|\ Y_k\geq L_k]+I\!E[f(n)-f(n-Y_k)|\ Y_k<L_k])^2}{f(n)+I\!E[f(n-Y_k)]}\\
&\quad\leq 2\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\tfrac{(I\!E[f(n)-f(n-Y_k)|\ Y_k\geq L_k])^2}{f(n)+I\!E[f(n-Y_k)]}\\
&\qquad\qquad +2\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\tfrac{(I\!E[f(n)-f(n-Y_k)|\ 0<Y_k<L_k])^2}{f(n)+I\!E[f(n-Y_k)]}.
\end{aligned} \tag{8}$$

The first term of (8) is

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\tfrac{(I\!E[f(n)-f(n-Y_k)|\ Y_k\geq L_k])^2}{f(n)+I\!E[f(n-Y_k)]}$$

$$\le \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} \frac{\mathbb{E}[f(n)+f(n-Y_k)\mid Y_k\ge L_k]\mathbb{E}[f(n)+f(n-Y_k)\mid Y_k\ge L_k]}{f(n)+\mathbb{E}[f(n-Y_k)]}$$
$$\le \sum_{k=1}^{\infty}\sum_{n=0}^{\infty} \mathbb{E}\left[f(n)+f(n-Y_k)\mid Y_k\ge L_k\right]$$
$$= 2\sum_{k=1}^{\infty}\mathbb{P}(Y_k\ge L_k),$$

and the second term of (8) is

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{(\mathbb{E}[f(n)-f(n-Y_k)\mid 0<Y_k<L_k])^2}{f(n)+\mathbb{E}[f(n-Y_k)]}$$
$$\le \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{\left(\sum_{0<m<L_k}(f(n)-f(n-m))g_k(m)\right)^2}{f(n)}$$
$$\le \sum_{k=1}^{\infty}\sum_{n=0}^{\infty}\frac{\left(\sum_{0<m<L_k}(f(n)-f(n-m))^2g_k(m)\right)\left(\sum_{0<m<L_k}g_k(m)\right)}{f(n)}$$
$$\le \sum_{k=1}^{\infty}\left\{\sum_{0<m<L_k}\left(\sum_{n=0}^{\infty}\frac{(f(n)-f(n-m))^2}{f(n)}\right)g_k(m)\right\}\left(\sum_{0<m<L_k}g_k(m)\right)$$
$$\le \sum_{k=1}^{\infty}\sup_{1\le\ell<L_k} J(\mathbf{X};\ell)\left(\sum_{0<m<L_k}g_k(m)\right)\left(\sum_{0<m<L_k}g_k(m)\right)$$
$$= \sum_{k=1}^{\infty}\sup_{1\le\ell<L_k} J(\mathbf{X};\ell)\left(\sum_{0<m<L_k}g_k(m)\right)^2$$
$$= \sum_{k=1}^{\infty}\sup_{1\le\ell<L_k} J(\mathbf{X};\ell)\mathbb{P}(0<Y_k<L_k)^2.$$

This completes the proof.

4 The case where X is geometrically distributed

Proof of Theorem 3. Since the proof is similar to that of [2], we shall only show the outline here. First we have

$$Z_k(n)=\sum_{m=0}^{n}q^{-m}g_k(m)-1=\mathbb{E}\left[q^{-Y_k}:Y_k\le n\right]-1$$

and $\rho(n) = q$, so that $Z_k(n)$ is non-decreasing and γ_k is given by (6). By Lemma 8, $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ is equivalent to (KS.1) and (KS.2).

Assume (KS.1). We have

$$\begin{aligned} & \mathbb{E}[Z_k(X_k) : Z_k(X_k) \geq 1] \\ = & \sum_{n=\gamma_k}^{\infty} pq^n \left(\mathbb{E}[q^{-Y_k} : Y_k \leq n] - 1\right) \\ = & q^{\gamma_k}\mathbb{E}[q^{-Y_k} : Y_k \leq \gamma_k] + \mathbb{P}(Y_k > \gamma_k) - q^{\gamma_k} \end{aligned}$$

and we have $\mathbb{E}[q^{-Y_k} : Y_k \leq \gamma_k] \geq 2$ if $\gamma_k < \infty$, hence

$$\mathbb{E}[Z_k(X_k) : Z_k(X_k) \geq 1] \geq \mathbb{P}(Y_k > \gamma_k) + q^{\gamma_k},$$

so that we have

$$\sum_{k=1}^{\infty} \mathbb{P}(Y_k > \gamma_k) + \sum_{k=1}^{\infty} q^{\gamma_k} = \sum_{k:\gamma_k<\infty} \mathbb{P}(Y_k > \gamma_k) + \sum_{k:\gamma_k<\infty} q^{\gamma_k} < \infty.$$

Furthermore, since $q^{-Y_k} \geq 1$ a.e., we have

$$\mathbb{E}[Z_k(X_k) : Z_k(X_k) \geq 1] \geq (1 - q^{\gamma_k})\mathbb{P}(\gamma_k \leq Y_k).$$

Since $\gamma_k \to \infty$ as $k \to \infty$ by (4), (KS.1) implies (3) and (4).

Conversely, assume (3) and (4). We have $\mathbb{E}[q^{-Y_k} : Y_k < \gamma_k] \leq 2$ by definition, so that we have

$$\mathbb{E}[Z_k(X_k) : Z_k(X_k) \geq 1] \leq q^{\gamma_k} + \mathbb{P}(Y_k \geq \gamma_k),$$

therefore (3) and (4) implies (KS.1). Consequently, (KS.1) is equivalent to (3) and (4).

Next, assume (KS.1), and denote by $\{Y_k'\}$ an independent copy of $\{Y_k\}, k \geq 1$. Then we have

$$\begin{aligned} & \mathbb{E}[Z_k(X_k)^2 : Z_k(X_k) < 1] \\ = & \sum_{n=0}^{\gamma_k - 1} pq^n \left(\mathbb{E}[q^{-Y_k} : Y_k \leq n] - 1\right)^2 \\ = & p\sum_{n=0}^{\gamma_k - 1} q^n \left(\mathbb{E}[q^{-(Y_k+Y_k')} : Y_k, Y_k' \leq n] - 2\mathbb{E}[q^{-Y_k} : Y_k \leq n] + 1\right) \end{aligned}$$

$$= \mathbb{E}[q^{-(Y_k\wedge Y_k')} : Y_k, Y_k' < \gamma_k] - \mathbb{P}(Y_k < \gamma_k) - q^{\gamma_k}\mathbb{E}[q^{-Y_k} : Y_k < \gamma_k]^2 \\ +\mathbb{P}(Y_k = \gamma_k) + \mathbb{P}(Y_k > \gamma_k) + 2q^{\gamma_k}\mathbb{E}[q^{-Y_k} : Y_k < \gamma_k] - q^{\gamma_k},$$

where $a \wedge b := \min\{a, b\}$. By $\mathbb{E}[q^{-Y_k} : Y_k < \gamma_k] \leq 2$ and the assumption, the last five terms in the final expression are summable. Therefore, under (KS.1), (KS.2) is equivalent to

$$\sum_{k=1}^{\infty}\left\{\mathbb{E}[q^{-(Y_k\wedge Y_k')} : Y_k, Y_k' < \gamma_k] - \mathbb{P}(Y_k < \gamma_k)\right\} < \infty.$$

Furthermore, since

$$\mathbb{P}(Y_k < \gamma_k) = \mathbb{P}(Y_k < \gamma_k, Y_k' < \gamma_k) + \mathbb{P}(Y_k < \gamma_k, Y_k' \geq \gamma_k),$$

we have

$$\sum_{k=1}^{\infty}\left\{\mathbb{E}[q^{-(Y_k\wedge Y_k')} : Y_k, Y_k' < \gamma_k] - \mathbb{P}(Y_k < \gamma_k)\right\} \\ = \sum_{k=1}^{\infty}\mathbb{E}[q^{-(Y_k\wedge Y_k')} - 1 : Y_k, Y_k' < \gamma_k] \\ -\sum_{k=1}^{\infty}\mathbb{P}(Y_k < \gamma_k)\mathbb{P}(Y_k' \geq \gamma_k),$$

where the second sum in the right-hand side is finite by (3). Thus under (KS.1), (KS.2) is equivalent to

$$\sum_{k=1}^{\infty}\mathbb{E}[q^{-(Y_k\wedge Y_k')} - 1 : Y_k, Y_k' < \gamma_k] < \infty,$$

which is expressed as

$$\sum_{k=1}^{\infty}\sum_{n=0}^{\gamma_k-1} q^{-n}\mathbb{P}(n < Y_k < \gamma_k)^2 < \infty,$$

because

$$\mathbb{E}[q^{-(Y_k\wedge Y_k')} - 1 : \ Y_k, Y_k' < \gamma_k] \\ = \mathbb{E}[\frac{p}{q}\sum_{m=0}^{\infty} q^{-m}\mathbf{I}_{[0,(Y_k\wedge Y_k')-1]}(m) : Y_k, Y_k' < \gamma_k]$$

Therefore (9) implies (3), (4) and (5).

(ii) By the assumption, $-a_k \geq -L$ for some $L > 0$ so that $q^{-a_k} \leq q^{-L}$. Hence the second term of (9) is equivalent to

$$\sum_{p_k(q^{-a_k}-1)\leq 1} p_k^2 < \infty.$$

On the other hand, if $p_k \to 0$, then since $a_k \leq L$, we have $p_k(q^{-a_k}-1) \to 0$. Hence $\#\{k \mid p_k(q^{-a_k}-1) > 1\} < \infty$. Therefore we obtain the assertion.

Next corollary is the case where $\mathbf{Y}$ also obeys a geometric distribution. Remark that $\sum_k q_k^2 = \sum_k I\!P(Y_k > 0)^2$ in this case.

Corollary 11. Assume $\mathbf{X}$ obeys a geometric distribution with $f(n) = \frac{1}{2^{n+1}}$, $n \in \mathbf{N}_0$, and $\mathbf{Y}$ also obeys a geometric distribution with $g_k(m) = p_k q_k^m$ $n \in \mathbf{N}_0$, $p_k, q_k > 0$, $p_k + q_k = 1$, $k \geq 1$. Then $\mu_{\mathbf{X}+\mathbf{Y}} \sim \mu_{\mathbf{X}}$ if and only if

$$\sum_{k=1}^{\infty} q_k^2 < \infty.$$

Proof. We have

$$\begin{aligned} Z_k(X_k)(n) &= I\!E[2^{Y_k} : Y_k \leq n] = \sum_{m=0}^{n} 2^m p_k q_k^m \\ &= \begin{cases} \dfrac{(1-q_k)\left\{(2q_k)^{n+1}-1\right\}}{2q_k - 1}, & q_k \neq \dfrac{1}{2}, \\ \dfrac{n+1}{2}, & q_k = \dfrac{1}{2}, \end{cases} \end{aligned}$$

so that

$$\gamma_k = \begin{cases} \infty, & q_k \leq \frac{1}{3} \\ \left[\dfrac{\log \frac{3q_k-1}{1-q_k}}{\log 2q_k}\right], & \frac{1}{3} < q_k < 1, q_k \neq \frac{1}{2} \\ 2, & q_k = \frac{1}{2}. \end{cases}$$

By (3) and (4), we obtain

$$\sum_k \sum_{m=\gamma_k}^{\infty} p_k q_k^m = \sum_k q_k^{\gamma_k} < \infty \tag{10}$$

and

$$\sum_k \left(\frac{1}{2}\right)^{\gamma_k} < \infty. \tag{11}$$

Next we consider the condition (5).

(i) The case where $\gamma_k < \infty$. We have

$$\begin{aligned}
\sum_k \sum_{m=0}^{\gamma_k - 1} 2^m \left\{ \sum_{z=m+1}^{\gamma_k - 1} (1-q_k) q_k^z \right\}^2 &= \sum_k \sum_{m=0}^{\gamma_k - 1} 2^m (q_k^{m+1} - q_k^{\gamma_k})^2 \\
&= \sum_k \sum_{m=0}^{\gamma_k} 2^m (q_k^{m+1} - q_k^{\gamma_k})^2 \geq \sum_k (q_k - q_k^{\gamma_k})^2 \\
&= \sum_k (q_k^2 - 2q_k^{\gamma_k+1} + q_k^{2\gamma_k}),
\end{aligned}$$

and by (10), (5) implies $\sum_k q_k^2 < \infty$.

Conversely, assume $\sum_k q_k^2 < \infty$, then there exists an N_1 such that $2q_k < \frac{1}{2}$ for any $k \geq N_1$. We have

$$\begin{aligned}
&\sum_{k=N_1}^{\infty} \sum_{m=0}^{\gamma_k - 1} 2^m (q_k^{m+1} - q_k^{\gamma_k})^2 = \sum_{K=N_1}^{\infty} \sum_{m=0}^{\gamma_k - 1} 2^m q_k^{2m+2} (1 - q_k^{\gamma_k - m - 1})^2 \\
\leq\ & \sum_{k=N_1}^{\infty} \sum_{m=0}^{\gamma_k - 1} 2^m q_k^{2m} q_k^2 = \sum_{k=N_1}^{\infty} \frac{q_k^2 (1 - (2q_k)^{\gamma_k})}{1 - 2q_k} \\
<\ & 2 \sum_{k=N_1}^{\infty} q_k^2 < \infty,
\end{aligned}$$

hence $\sum_k q_k^2 < \infty$ implies (5). Therefore (5) is equivalent to $\sum_k q_k^2 < \infty$.

(ii) The case where $\gamma_k = \infty$. We have

$$\sum_k \sum_{m=0}^{\gamma_k - 1} 2^m \left\{ \sum_{z=m+1}^{\gamma_k - 1} (1-q_k) q_k^z \right\}^2 = \sum_k \sum_{m=0}^{\infty} 2^m q_k^{2(m+1)} = \frac{q_k^2}{1 - 2q_k^2}$$

and

$$\frac{9}{7} q_k^2 < \frac{q_k^2}{1 - 2q_k^2} < q_k^2,$$

then it also holds that (5) is equivalent to $\sum_k q_k^2 < \infty$.

If $\sum_k q_k^2 < \infty$, then $q_k \to 0$, so that there exists an N_2 such that $q_k \leq \frac{1}{3}$ for $k \geq N_2$, so that $\gamma_k = \infty$ for $k \geq N_2$. Hence $\sum_k q_k^2 < \infty$ implies $\sum_k q_k^{\gamma_k} < \infty$ and $\sum_k 2^{-\gamma_k} < \infty$.

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