Subjective evaluation based on distorted probability

Aoi Honda, Takafumi Nakano and Yoshiaki Okazaki
Kyushu Institute of Technology

Dept. of Control Engineering and Science,
Fac. of Computer Science and Systems Engineering
Kyushu Institute of Technology, Kawazu, Iizuka 820-8502, JAPAN
email: aoi@ces.kyutech.ac.jp

Abstract— **The measure of subjective evaluation be considered as a fuzzy measure on the finite set of items. One can observe the fuzzy measure only for a small family of evaluative events. Therefore it is necessary to estimate the fuzzy measure for the family of all evaluative events (identification of fuzzy measure).**

In this paper we discuss a classification of the finite fuzzy measures and a method of estimation of a fuzzy measure using distorted probability.

Keywords: distorted probability; subjective evaluation; identification of fuzzy measure; classification of distorted probability

I. Introduction

Fuzzy measure is a non-additive set function and is recently applied toward the evaluation problems. The measure of each person's subjective evaluation is not additive in a practical problems , and the procedure piling up the partial or local value judgment is not linear . Hence the additive measure and the linear integration are not necessarily adequate in the subjective evaluation problems. The fuzzy measure (the fuzzy integration) is considered as a basis formulating the actual evaluation process.

In ordinary case, we can only observe a small family of events and we can obtain the data of the fuzzy measure for these small class of events. So we have to identify the whole fuzzy measure starting from the observed small data. In the case of an additive measure, we can reconstruct the whole measure from the measures of elementary events (events consisted of one item). But the fuzzy measure is not completely determined by the data of elementary events.

There are several proposals identifying the fuzzy measure . The identification procedure is as follows. Let $M = \{g_\lambda | \lambda \in \Lambda\}$be a class of fuzzy measures parametrized by Λ and let $\mathcal{A}$ be the small class of events in which the value of fuzzy measure $g(A), A \in \mathcal{A}$, is known. Then the problem is to find the best parameter $\lambda \in \Lambda$ and g_λ which estimates g appropriately . For example , it is proposed the methods using the class of λ-fuzzy measures , the class of χ-fuzzy measures and so on.

In this paper we shall introduce the notions of the distortion, the type and the invariance of fuzzy measures, and present an approach for identifying a fuzzy measure using the class of distorted fuzzy measures . We shall also examine the examples comparing with other methods.

II. Preliminary

In this section we shall introduce a distorted measure , a type of the fuzzy measure, and discuss their properties. Throughout this paper, let X be a nonempty finite set, n be the number of elements in X and $\mathcal{P}(X)$ be the power set of X.

Definition 1 A set function $g : \mathcal{P}(X) \to [0,\ 1]$ is called a fuzzy measure iff

1) $g(\emptyset) = 0,\ g(X) = 1$(normalization)
2) $A, B \in \mathcal{P}(X)$, and $A \subset B$ imply $g(A) \leq g(B)$(monotonicity).

The set $A \in \mathcal{P}(X)$ is called an event and $g(A)$ is the magnitude of the event A The additivity is not supposed here, and we can not use the results of the standard measure theory. Now we shall introduce a concept of the distortion.

Definition 2 For two fuzzy measures g, h on $\mathcal{P}(X)$, if there exists a non-decreasing function $f : [0,\ 1] \to [0,\ 1]$ satisfying that for any set $A \in \mathcal{P}(X)$

$$g(A) = f \circ h(A) = f(h(A))$$

, then g is called a distortion of h (g is distorted to h).

f is called scaling function. We can regard g as a new measure changing the scale of h-measure through the scaling function f. Because f is non-decreasing, the magnitude relation is descend from h to g. If h is a probability measure, g would be expected to have good properties arising from the additivity of h.

Definition 3 A fuzzy measure $\Pi : \mathcal{P}(X) \to [0,\ 1]$ is called a possibility measure iff

there exists a function $\pi : X \to [0,\ 1]$ such that

$$\sup\{\pi(x) | x \in X\} = 1$$

and it holds that

$$\Pi(A) = \sup\{\pi(x) | x \in A\} \quad \forall A \subset X$$

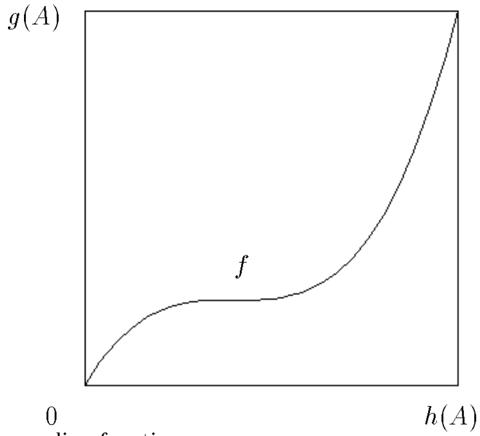

Fig. 1. scaling function

$$where \quad \sup \pi(\emptyset) = 0.$$

Definition 4 A fuzzy measure $g_\lambda : \mathcal{P}(X) \to [0,\ 1]$ is called a λ-fuzzy measure iff
it holds that

$$g_\lambda(A \cup B) = g_\lambda(A) + g_\lambda(B) + \lambda g_\lambda(A) g_\lambda(B), \qquad (1)$$
$$-1 < \lambda < \infty$$

for any $A, B \in \mathcal{P}(X)$.

The property (1) is calld λ-additivity.

Definition 5 A fuzzy measure $g_\chi : \mathcal{P}(X) \to [0,\ 1]$ is called a χ-fuzzy measure iff
for any $A \in \mathcal{P}(X)$

$$g_\chi(A) = \chi^{|A|-1}(\sum_{x_i \in A} g_\chi(\{x_i\})), \qquad (2)$$
$$\chi \geq 1 - \frac{\min_{x_i \in X} g(\{x_i\})}{\sum_{x_i \in X} g(\{x_i\})}$$

where $|A|$ is the number of elements in A

The χ-fuzzy measures also have the property close to additivity.

Next we shall define the type that is fundamental in the classification of the fuzzy measure. Let Γ be the set of all permutations of $\mathcal{P}(X)$ elements, and γ be a element of Γ

$$\Gamma := \{(A_1, A_2, \ldots, A_{2^n}) | \{A_1, A_2, \ldots, A_{2^n}\} = \mathcal{P}(X)\}$$

$$\gamma \in \Gamma, A_1, \ldots, A_{2^n} \in \mathcal{P}(X)$$

It holds that $|\Gamma| = 2^n!$.

Definition 6 Let $\gamma = (A_1, A_2, \ldots, A_{2^n}) \in \Gamma$. A set function $g : \mathcal{P}(X) \to [0,\ 1]$ is called of type $\gamma(\gamma \in \Gamma)$ iff

$$g(A_1) \leq g(A_2) \leq \cdots \leq g(A_{2^n}).$$

Definition 7 Let $\gamma = (A_1, A_2, \ldots, A_{2^n}) \in \Gamma$. A set function $g : \mathcal{P}(X) \to [0,\ 1]$ is called of strictly type $\gamma(\gamma \in \Gamma)$ iff

$$g(A_1) < g(A_2) < \cdots < g(A_{2^n})$$

Theorem 1 *Let $\mathcal{S}(\gamma)$ be the set of all strictly type γ fuzzy measures. If a fuzzy measure $g_0 \in \mathcal{S}(\gamma)$ then any fuzzy measure $g \in \mathcal{S}(\gamma)$ is distorted to g_0. Particularly, $\mathcal{S}(\gamma) = \{f \circ g_0 | f : [0,\ 1] \to [0,\ 1]$ is non-decreasing and $f(0) = 0, f(1) = 1\}$.*

Theorem 2 *If the fuzzy measures g, g' satisfy following conditions, g is distorted to g'.*
1) *g, g' are both of type γ*
2) *For any $A, B \in \mathcal{P}(X)$*
$g'(A) = g'(B) \Rightarrow g(A) = g(B), \quad \forall A, B \in \mathcal{P}(X)$.

Proof Let f be the piecewise linear function which link the points $(g'(A_i), g(A_1)), i = 1, 2, \ldots$. It holds that $g(A_i) = f(g'(A_i)), i = 1, 2, \ldots$. The function f can be defined without contradiction because of the assumption 2) , and f is non-decreasing.

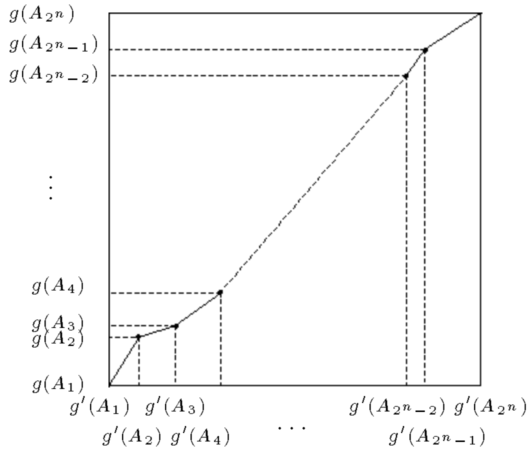

Fig. 2. The scaling function of theorem 2

If g is a fuzzy measure, then the set function g^d defined by

$$g^d(A) = 1 - g(A^c)$$

is called a dual measure.

Theorem 3 *If a fuzzy measure g and its dual measure are of same type , then it holds the next condition, and vice versa.*

$$g(A) \leq g(B) \Rightarrow g(A^c) \geq g(B^c)$$

The following theorem is important.

Theorem 4 *Let g, g' be fuzzy measures. If for any $A, B \in A \in \mathcal{P}(X)$*

$$g'(A) \leq g'(B) \Rightarrow g(A) \leq g(B),$$

then g is distorted to g'.

Corollary 1 Let g, g' be fuzzy measures. If for any $A, B \in A \in \mathcal{P}(X)$

$$g'(A) \leq g'(B) \iff g(A) \leq g(B),$$

then g is distorted to g' and there exists a scaling function f is which is monotone increasing.

A. distorted probability

Plainly speaking the distorted probability is a fuzzy measure distorted to the linear measure.

Definition 8 $P : \mathcal{P}(X) \to [0,\ 1]$ is called a probability measure iff

1) $P(\emptyset) = 0,\ P(X) = 1$
2) For any $A, B \in \mathcal{P}(X)$, $A \cap B = \emptyset$ implies $P(A \cup B) = P(A) + P(B)$

The fuzzy measure g is distorted to the probability P iff for any $A, B \in \mathcal{P}(X)$ $P(A) \le P(B)$ implies $g(A) \le g(B)$, in other words the magnitude relation of the events A, B in the probability measure P is inherited in g.

The probability measure has the following property of consistency for the disjoint union.([2])

For any $A, B, C \in \mathcal{P}(X)$

$$P(A) \le P(B), A \cap C = B \cap C = \emptyset \Rightarrow P(A \cup C) \le P(B \cup C).$$

If g is distorted to the probability P,g have the same magnitude relation as P, so it holds that

$$g(A) \le g(B), A \cap C = B \cap C = \emptyset \Rightarrow g(A \cup C) \le g(B \cup C).$$

It follows from what has been shown that we should consider the magnitude relation of events. The concept of the type is introduced to see this relation .

Example 1 [1] The λ-fuzzy measure is a distorted probability.

Example 2 The possibility measure on the finite set is a distorted probability.

Fact 1 Let $X = \{x_1,\ x_2\}$ be a two point set, and $g : \mathcal{P}(X) \to [0,1]$ be a fuzzy measure satisfying $g(\{x_1\}) \le g(\{x_2\})$. Then , according to the order of the value $g(A)$ for each event $A \in \mathcal{P}(X)$, the fuzzy measures are classified into just one type, $\gamma_1 = \{\emptyset,\ \{x_1\},\ \{x_2\},\ X\}$.

In particular, all fuzzy measures are distorted probabilities.

Fact 2 Let $X = \{x_1,\ x_2,\ x_3\}$ be a three point set, and $g : \mathcal{P}(X) \to [0,1]$ be a fuzzy measure satisfying that $g(\{x_1\}) \le g(\{x_2\}) \le g(\{x_3\})$. Then , according to the order of the value $g(A)$ for each event $A \in \mathcal{P}(X)$, the fuzzy measures are classified into eight types as follows. The measures only in two types $\gamma_1,\ \gamma_2$ are the distorted probabilities.

$$\gamma_1 = \{\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}, \{x_3\}, \{x_1, x_3\}, \{x_2, x_3\}, X\}$$
$$\gamma_2 = \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, \{x_2, x_3\}, X\}$$
$$\gamma_3 = \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_3\}, \{x_1, x_2\}, \{x_2, x_3\}, X\}$$
$$\gamma_4 = \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2\}, X\}$$
$$\gamma_5 = \{\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}, \{x_3\}, \{x_2, x_3\}, \{x_1, x_3\}, X\}$$
$$\gamma_6 = \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_2, x_3\}, \{x_1, x_3\}, X\}$$
$$\gamma_7 = \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_2, x_3\}, \{x_1, x_2\}, \{x_1, x_3\}, X\}$$
$$\gamma_8 = \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_2, x_3\}, \{x_1, x_3\}, \{x_1, x_2\}, X\}$$

Fact 3 Let $X = \{x_1,\ x_2,\ x_3,\ x_4\}$ be a four point set, and $g : \mathcal{P}(X) \to [0,1]$ be a fuzzy measure satisfying that $g(\{x_1\}) \le g(\{x_2\}) \le g(\{x_3\}) \le g(\{x_4\})$. Then , according to the order of the value $g(A)$ for each event $A \in \mathcal{P}(X)$, the fuzzy measures are classified into 70016 types, of which 14 types are the one in which the distorted probability belong.

If X is a three point set, then the type of the λ-fuzzy measure is γ_1 or γ_2 and the type of the possibility measure is γ_1. If X is a four point set, then the type of the λ-fuzzy measure is γ_1 or $\ldots$ or γ_{14} and the type of possibility measure is just γ_{14}.

Fact 4 Let $X = \{x_1,\ x_2,\ x_3,\ x_4,\ x_5\}$ be five points set and , $g(\{x_1\}) \le g(\{x_2\}) \le g(\{x_3\}) \le g(\{x_4\}) \le g(\{x_5\})$. In this case, the distorted probability is classifiable into 514 types.

It follows from the four facts stated above that fuzzy measure has more degrees of freedom than probability measure.

III. Identification of the fuzzy measure

In this section, we shall make a new approach for the identification of fuzzy measures using the class of the distorted probabilities.

Let $X = \{x_1, x_2, \ldots, x_n\}$ be a n point set . On the assumption that fuzzy measure is known only on a subfamily $\mathcal{A} \subset \mathcal{P}(X)$. It is required that the value $g(\{x_i\})$ of each elementary event $\{x_i\}$ must be known in advance. We have to estimate the fuzzy measure $g(B)$ for the event B of unknown value, that is, for $B \in \mathcal{P}(X) \setminus \mathcal{A}$(Identification of Fuzzy Measure). We would like to minimize the residual sum of squares of estimates and true values .

A. Simple estimation

We shall propose a simple method of the identification, in the case where no other information, expect the monotonicity on g, is available. If we suppose only that g is a fuzzy measure, we know nothing but monotonicity about g. Unknown value $g(A), A \in \mathcal{P}(X) \setminus \mathcal{A}$ must satisfy

$$MIN \le g(A) \le MAX,$$

where $MIN = \max(g(B); A \supset B \in \mathcal{A}),\ MAX = \min(g(C); A \subset C \in \mathcal{A})$

Each value in the interval $[MIN,\ MAX]$ can be taken as an estimation of $g(A)$. It we minimize the residual sum of squares, the estimation should be $\frac{MAX+MIN}{2}$.

B. Proposed method–estimation using distorted probability measure

If we don't have any information about the data except for monotonicity, simple estimation is the best. But practically it might be expected that the data has the attributes of some kind. For example, people's subjectivity may have some tendency to stack simultaneously and additively the evaluations of different natures. Here we discuss a method of the identification confined to class of the distorted probability. The distorted probability preserves the magnitude relation of the probability measure, even though it doesn't have the additivity , as we have described in section II above. When we identify the fuzzy measure g assuming that g is a distorted probability, what we have to do is to find the probability measure P and the scaling function f with $g = f \circ P$.

This method consists of the three steps: (1) decision of the type of the fuzzy measure g from the known data, (2) decision of the probability measure P for each elementary event, and (3) selection of the scaling function f. First, we choose the type of the fuzzy measure g by the known data. If there are some candidates of type, we decide the adequate type using the magnitude relation of known data assuming that g satisfies the additivity (see also appendix B). We may give the probability P arbitrarily, but we present standard examples of probabilities in each class of the type in advance (table I). These probabilities are required to have the values $\{P(A); A \in \mathcal{P}(X)\}$ equally distributed in $[0, 1]$. For example , for type γ_{14} we set $P(x_1) = \frac{1}{15}, P(x_2) = \frac{2}{15}, P(x_3) = \frac{4}{15}, P(x_4) = \frac{8}{15}$, then $\{P(A)|A \in \mathcal{P}(X)\} = \{0, \frac{1}{15}, \frac{2}{15}, \frac{3}{15}, \frac{4}{15}, \frac{5}{15}, \frac{6}{15}, \frac{7}{15}, \frac{8}{15}, \frac{9}{15}, \frac{10}{15}, \frac{11}{15}, \frac{12}{15}, \frac{13}{15}, \frac{14}{15}, 1\}$, where $\mathcal{P}(X) = \{\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}, \{x_3\}, \{x_1, x_3\}, \{x_2, x_3\}, \{x_1, x_2, x_3\}, \{x_4\}, \{x_1, x_4\}, \{x_2, x_4\}, \{x_1, x_2, x_4\}, \{x_3, x_4\}, \{x_1, x_3, x_4\}, \{x_2, x_3, x_4, X\}$. The scaling function f is given as in Theorem 2 using only known events $A_i \in \mathcal{A}$. We can obtain the estimations of the unknown events from $g(A) = f \circ P(A)$.

TABLE I
THE GIVEN PROBABILITY FOR $X = \{x_1, x_2, x_3, x_4\}$

Type	$P(\{x_1\})$	$P(\{x_2\})$	$P(\{x_3\})$	$P(\{x_4\})$
γ_1	$\frac{5}{29}$	$\frac{7}{29}$	$\frac{8}{29}$	$\frac{9}{29}$
γ_2	$\frac{4}{23}$	$\frac{5}{23}$	$\frac{6}{23}$	$\frac{8}{23}$
γ_3	$\frac{3}{33}$	$\frac{8}{33}$	$\frac{10}{33}$	$\frac{12}{33}$
γ_4	$\frac{4}{28}$	$\frac{5}{28}$	$\frac{8}{28}$	$\frac{11}{28}$
γ_5	$\frac{2}{25}$	$\frac{6}{25}$	$\frac{7}{25}$	$\frac{10}{25}$
γ_6	$\frac{2}{19}$	$\frac{4}{19}$	$\frac{5}{19}$	$\frac{8}{19}$
γ_7	$\frac{2}{21}$	$\frac{4}{21}$	$\frac{5}{21}$	$\frac{10}{21}$
γ_8	$\frac{2}{23}$	$\frac{4}{23}$	$\frac{5}{23}$	$\frac{12}{23}$
γ_9	$\frac{2}{25}$	$\frac{4}{25}$	$\frac{9}{25}$	$\frac{10}{25}$
γ_{10}	$\frac{3}{25}$	$\frac{4}{25}$	$\frac{8}{25}$	$\frac{10}{25}$
γ_{11}	$\frac{2}{29}$	$\frac{6}{29}$	$\frac{9}{29}$	$\frac{12}{29}$
γ_{12}	$\frac{2}{25}$	$\frac{4}{25}$	$\frac{8}{25}$	$\frac{11}{25}$
γ_{13}	$\frac{2}{27}$	$\frac{4}{27}$	$\frac{8}{27}$	$\frac{13}{27}$
γ_{14}	$\frac{1}{15}$	$\frac{2}{15}$	$\frac{4}{15}$	$\frac{8}{15}$

C. Identification using λ-fuzzy measure

The fuzzy measure g is estimated in the class of the λ-fuzzy measures. In this method, to identify the fuzzy measure is reduced to find the best value of the parameter λ.

We choose the best value of λ which minimizes

$$\sum_{A \in \mathcal{A}} (g(A) - g_\lambda(A))^2.$$

We must solve the algebraic equation of λ of order $(n-1)^2$. If X is an n point set and $\mathcal{A}$ does not contain some one point sets , then first we shall estimate the measure of each one point set after deciding the value of λ, and estimate other events using them.

D. Identification using χ-fuzzy measure

The fuzzy measure g is estimated in the class of the χ-fuzzy measures.

In this method, to identify the fuzzy measure is reduced to find the best value of the parameter χ just like in the method of the λ-fuzzy measure.

We choose the rest value of χ which minimalizes

$$\sum_{A \in \mathcal{A}} (g(A) - g_\chi(A))^2.$$

We must solve the equation of χ of order $(n-1)^2$ In the case where X is an n point set and $\mathcal{A}$ does not contain some one point sets , the same procedure to the case of the λ-fuzzy measure is available.

IV. Case study

A. About the used data

We set following 2 questions.

1) (i) Select a concrete commodity of your interest and indicate four criterions $\{x_1, x_2, x_3, x_4\}$ for purchasing the best thing.
   (ii) Evaluate your purchasing level of the whole events , when all the criterion in the event are satisfactorily fulfilled.
2) (i) Select four entertainment talents $\{x_1, x_2, x_3, x_4\}$.
   (ii) Evaluate the degree of awaking your interest of the whole events, when you can see all the talents in the event all together in TV.

The maximum number of each marks for evaluation is 100.

We set $X = \{x_1, x_2, x_3, x_4\}$ and suppose that the following data $g(\{x_1\}), g(\{x_2\}), g(\{x_3\}), g(\{x_4\}), g(\{x_1, x_2\}), g(\{x_3, x_4\}), g(\{x_1, x_2, x_3\}), g(X)$ are the known data and the rest are used as the verification data. The estimation is done by using only the known data. The method of steepest decent is used in estimating parameters λ, χ of C and D. In the estimation using the distorted probability, first we determine the type of g by the known data. If the order of the known data is in classes below , we need not to modify the data.

$$k_1 = \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_4\}, \{x_1, x_2\}, \{x_3, x_4\}, \{x_1, x_2, x_3\}, X\}$$
$$k_2 = \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_4\}, \{x_1, x_2, x_3\}, \{x_3, x_4\}, X\}$$
$$k_3 = \{\emptyset, \{x_1\}, \{x_2\}, \{x_3\}, \{x_1, x_2\}, \{x_1, x_2, x_3\}, \{x_4\}, \{x_3, x_4\}, X\}$$
$$k_4 = \{\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}, \{x_3\}, \{x_4\}, \{x_1, x_2, x_3\}, \{x_3, x_4\}, X\}$$
$$k_5 = \{\emptyset, \{x_1\}, \{x_2\}, \{x_1, x_2\}, \{x_3\}, \{x_1, x_2, x_3\}, \{x_4\}, \{x_3, x_4\}, X\}$$

If the order of the known data is k_1, then the candidates of type are γ_1 and γ_2 (cf. theorem 3). And so the type is γ_1 when $g(\{x_1\}) + g(\{x_4\}) \leq g(\{x_2\}) + g(\{x_3\})$, otherwise the type is γ_2. If the order of the known data is not in the classes $k_1, \ldots, k_5$ disagree all of $k_1, \ldots, k_5$, then it would be modified such as the modified order belongs to one of the classes $k_1, \ldots, k_5$ by way in the section V-A. We discribe further details about the method of modification in appendix B.

B. Result and examination

Table II shows the averages and variances of the residual sum of squares of estimates and true values for methods A, B, C and D. Err_1 is the residual sum of squares of estimates in the case of replacing the known data by the estimated values, and Err_2 is in the case of using the known data not replacing. Remark that in the estimation process, we can recalculate the values of $g(A)$ for a known event A. In general, the recalculated value is different of the originally known data. So we consider these two errors. Our method is advantageous in Err_1, because the known data is same as estimated one when the modification is unnecessary.

$$Err_1 = \frac{1}{N} \sum_{i=1}^{N} \sum_{A \in \mathcal{P}(X)} (g(A) - \bar{g}(A))^2$$
$$Err_2 = \frac{1}{N} \sum_{i=1}^{N} \sum_{A \in \mathcal{P}(X) \setminus \mathcal{A}} (g(A) - \bar{g}(A))^2$$

where N is a number of data : 20

In the 20 times identifications, the numbers of the best estimations with respect to Err_1 are : distorted probability = 15, χ-fuzzy measure = 3, λ-fuzzy measure = 1 and the simple estimation = 1. In Err_2, the numbers of the best estimations are: distorted probability = 13, χ-fuzzy measure = 6, λ-fuzzy measure = 1.

In the method of the distorted probability, there were cases that the estimation were far from the true data when the decision of the type is wrong , but the estimation was better than other methods even in this case . In the case where the type is decided correctly or the type is similar, the estimations are very close to the true value. (On the similarity of type, see appendix A)

TABLE II

THE AVERAGES AND VARIANCES OF RESIDUAL SUM OF SQUARES OF ESTIMATES AND TRUE VALUES

Method		Simple	Distorted	λ-fuzzy	χ-fuzzy
Err_1	residual	0.1189	0.0390	0.0837	0.0871
	variance	0.0081	0.0006	0.0023	0.0030
Err_2	residual	0.1189	0.0386	0.0531	0.0549
	variance	0.0081	0.0006	0.0010	0.0012

V. HUMAN SUBJECTIVITY

In this section we will examine the four fuzzy measures if they are close to the human subjectivity or not, using the above data.

A. distorted probability

All collected data satisfy monotonicity, so they are fuzzy measures. First we examine the distorted probability. For each data we will modify them to one of the 14 type of the distorted probability measures. (See appendix B for a full account of the manner of the modifications.) The minimum value of the residual sum of squares of data between before and after modifications

$$E_P = \sum_{A \in \mathcal{P}(X)} (g(A) - g_P(A))^2$$

among 14 types, is the error of the original data and distorted probability. The type of the data is determined as the type which minimizes the residual sum of squares. The average and variance of the residual sum of squares are 0.0058 and 0.0000. There was one datum which is avoided the modification, that is, there was one distorted probability among 20 data. Next we will estimate by the simple method and by the distorted probability on the assumption that one of the data is unknown. Since X is 4 point set, so one set of data have 16 values of which we will estimate for 14 values other than empty and whole sets, on the assumption that one value is unknown. Table IV shows the averages of the errors E_2:

$$E_2 = \frac{1}{20} \sum_{i=1}^{20} \sum_{A \in \mathcal{P}(X)} (g(A) - \bar{g}(A))^2.$$

B. λ-fuzzy measure

For the 20 sets of data, we obtain best estimate of λ which minimizes the residual sum of squares between the estimations and true values. Table III shows the averages of errors E_λ:

$$E_\lambda = \sum_{A \in \mathcal{P}(X)} (g(A) - g_\lambda(A))^2.$$

C. χ-fuzzy measure

We will obtain χ in the same way as λ-fuzzy measure and calculate the error :

$$E_\chi = \sum_{A \in \mathcal{P}(X)} (g(A) - g_\chi(A))^2.$$

D. Result and examination

TABLE III

E_2 Simple estimation and Distorted probability

Method	Simple estimation	Distorted probability
Average	0.1182	0.0726
Variance	0.0071	0.0076

TABLE IV

E_2 λ-fuzzy measure and χ-fuzzy measure

Method	λ-fuzzy measure	χ-fuzzy measure
Average	0.0784	0.1323
Variance	0.0021	0.0630

As the table III indicates, the average and variance of errors by the method of the distorted probability are smaller than the one by simple method. We may, therefore, reasonably conclude that the original data is closer to the distorted probability than the simply estimated fuzzy measure. In addition they are both small as seen by the table III. We may say that these conclusions are in some sense very natural since the degree of freedom of the distorted probability is very high, but the identification may be complicated in some part. Thus it is concluded that the data treated here be regarded as the distorted probability as opposed to the λ-fuzzy measure and to the χ-fuzzy measure . Furthermore the number of the best estimation is 14 in the distorted probability, 4 in the χ-fuzzy measure, 2 in the λ-fuzzy measure ,1 in the simple estimation.

VI. Concluding remarks for further study

We proposed an estimation of the fuzzy measure and our class of the distorted probability seems suitable for considering the human subjectivities. If X is $5, 6, \ldots$ point set, then we have to determine all types of probability measures. It will be possible but unrealistic. It seems reasonable to choose some of typical types.

In this time, we had sent out questionnaires grading viscerally. According to the impression of the collaborator, this is difficult to answer and the pair comparison method is much easier to answer. In the pair comparison method we have to compare $105(= \frac{2^n \times (2^n - 1)}{2})$ times, therefore it should compare a part of them efficiently. And we need to identify fuzzy measure of the rest.

References

[1] Rudolf Kruse, "A note on λ- additive fuzzy measures". *Fuzzy Sets and Systems*, Vol. 8, 1982.

[2] T. L. Fine, *Theories of Probability* ,Academic Press, New York,1982.

[3] Isamu Okada and Hiroyuki Tajima, *Decision making with χ-fuzzy measure in hierarchical structure*(in japanese) , 12th Fuzzy System Symposium, pp875–878, Japan Society for Fuzzy and Systms, 1996.

Appendix

A. On the similarity of the type

We shall introduce the concept of the distance between two types.

Definition 9 Let γ_1 and γ_2 be lists, and Λ be entirety of division of $\gamma_1 and \gamma_2$ such that $x \in B_i \iff x \in B'_i$,in other words the entirety of $\gamma_1 = \{B_1, \ldots, B_m\}, \gamma_2 = \{B'_1, \ldots, B'_m\}$. We shall define the distance between γ_1 and γ_2 by:

$$D(\gamma_1, \gamma_2) = \inf_{\Lambda} \left\{ \sqrt{\sum_{B_i \neq B'_i} |B_i|^2} \right\}$$

where $|B_i|$ is the number of elements in B_i.

If $D(\gamma_1, \gamma_2)$ is small , then we may consider that γ_1 and γ_2 are similar.

Theorem 5 *$D(\gamma_1, \gamma_2)$ satisfy the axiom of the distance.*

Example Let $x_1 = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}, x_2 = \{1, 3, 2, 4, 5, 8, 7, 6, 9, 10\}$ and $x_3 = \{1, 2, 4, 5, 6, 3, 7, 8, 9, 10\}$, then

$$\begin{aligned} D(x_1, x_2) &= \sqrt{|\{2, 3\}|^2 + |\{6, 7, 8\}|^2} = \sqrt{13} \\ D(x_2, x_3) &= \sqrt{|\{3, 2, 4, 5, 8, 7, 6\}|^2} = 7 \\ D(x_1, x_3) &= \sqrt{|\{3, 4, 5, 6\}|^2} = 4 \end{aligned}$$

B. The way of modification

Let $B = \{B_1, \ldots, B_m\}$ and $B' = \{B'_1, \ldots, B'_m\}$ be divisions of γ and γ' which attain the value $D(\gamma, \gamma')$. We replace all elements in B_i such that $B_i \neq B'_i$ average value of the elements in B_i.

Example The case of modifying

$$\{g(C), g(B), g(A), g(E), g(D)\}$$

to

$$\{g'(A), g'(B), g'(C), g'(D), g'(E)\}.$$

The division is $\{B_1, B_2\} = \{\{g(C), g(B), g(A)\}, \{g(E), g(D)\}\}$ and $\{B'_1, B'_2\} = \{\{g'(A), g'(B), g'(C)\}, \{g'(D), g'(E)\}\}$, then

$$g(C) = g(B) = g(A) = \frac{g(C) + g(B) + g(A)}{3},$$

$$g(E) = g(D) = \frac{g(E) + g(D)}{2}.$$

In section III.B and section V.A we have modified to the type of minimal error comparing to the original type.