Distortion of Fuzzy Measures

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Abstract— **Let g and μ be fuzzy measures on the σ-algebra $\mathcal{B}$. g is said to be a distortion of μ if there is a non-decreasing function $f : [0,\infty] \to [0,\infty]$ satisfying that $g(E) = f(\mu(E)), E \in \mathcal{B}$. The general condition for the distortion between two fuzzy measures are discussed. The distortion can be characterize by the strong invariance. The fuzzy measure g is called strongly μ-invariant if $\mu(A) \leq \mu(B)$ implies that $g(A) \leq g(B)$ for every $A, B \in \mathcal{B}$. Then g is a distortion of μ if and only if g ia strongly μ-invariant. We give a characterization of the possibility measure a distortion of a σ-additive measure.**

I. Introduction and Preliminaries

We shall discuss the distortion property between fuzzy measures. The class of distorted fuzzy measure was introduced and investigated by Chateauneuf[1], Schmeidler[4] and Yaari[6], see also Pap[3] and Wang and Klir[5].

Let X be a set and $\mathcal{B}$ be a σ-algebra of subsets of X, that is, $\mathcal{B}$ satisfy the following conditions:

1) $\phi \in \mathcal{B}$,
2) if $E \in \mathcal{B}$, then $E^c \in \mathcal{B}$, and
3) if $E_n \in \mathcal{B}(n = 1, 2, ...)$, then $\cup_{n=1}^{\infty} E_n \in \mathcal{B}$.

Definition 1. The set function $g : \mathcal{B} \to [0,\infty]$ is called the fuzzy measure if the following conditions are satisfied:

1) $g(\phi) = 0$, and
2) for every $A, B \in \mathcal{B}$ with $A \subset B$, it follows that $g(A) \leq g(B)$(the monotonicity).

Definition 2. The set function $\mu : \mathcal{B} \to [0,\infty]$ is called the σ-additive measure if the following conditions are satisfied:

1) $\mu(\phi) = 0$, and
2) for every pairwise disjoint family $E_n \in \mathcal{B}(n = 1, 2, ...)$, it follows that $\mu(\cup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} E_n$(the σ-additivity).

Definition 3. The function $f : [0,\infty] \to [0,\infty]$ is called the scaling function (or the distortion function) if the following conditions are satisfied:

1) $f(0) = 0$, $f(\infty) = \infty$, and
2) f is non-decreasing.

Let f be a distortion function and $h : \mathcal{B} \to [0,\infty]$ be a fuzzy measure. Then the set function $g(A) = f(h(A)), A \in \mathcal{B}$, is also a fuzzy measure. We denote this fuzzy measure g by $g = f \circ h$.

Definition 4. Let g and h be two fuzzy measures on the σ-algebra $\mathcal{B}$. Then g is said to be a distortion of h if there exists a distortion function f satisfying that $g = f \circ h$.

Definition 5. Let g and h be two fuzzy measures on the σ-algebra $\mathcal{B}$. Then g is said to be h-invariant if $h(A) = h(B), A, B \in \mathcal{B}$, implies that $g(A) = g(B)$.

Definition 6. Let g and h be two fuzzy measures on the σ-algebra $\mathcal{B}$. Then g is said to be strongly h-invariant if $h(A) \leq h(B), A, B \in \mathcal{B}$, implies that $g(A) \leq g(B)$.

Definition 7. The set function $\Pi : \mathcal{B} \to [0, 1]$ is called the possibility measure if it holds that $\Pi(A \cup B) = \Pi(A) \vee \Pi(B) = \max\{\Pi(A), \Pi(B)\}$ for every $A, B \in \mathcal{B}$.

Definition 8. The fuzzy measure g is continuous from below if for every increasing sequence $A_n \uparrow A, A_n, A \in \mathcal{B}$, it follows that $g(A_n) \uparrow g(A)$.

Definition 9. The fuzzy measure g is said to be completely additive if for every directed family $\mathcal{A} = \{A_\alpha | \alpha \in I\} \subset \mathcal{B}$ increasing to A(that is, for each $A_\alpha, A_\beta \in \mathcal{A}$, there exists $A_\gamma \in \mathcal{A}$ such that $A_\alpha \cup A_\beta \subset A_\gamma$, and $\cup_{\alpha \in I} A_\alpha = A$), is follows that $g(A_\alpha) \uparrow g(A)$.

II. Invariance and Distortion

It is clear that if $g = f \circ h$ for a distortion function f, then g is strongly h-invariant, and also h-invariant. We shall consider the converse.

Theorem 1(Honda and Okazaki[2]). Let g and h be two fuzzy measures on the σ-algebra $\mathcal{B}$. Then g is a distortion of h if and only if g is strongly h-invariant.

Proof. The necessity is clear. We show the converse. We set the ranges $R(g), R(h)$ of g, h, respectively, by $R(g) = \{g(A) : A \in \mathcal{B}\}$ and $R(h) = \{h(A) : A \in \mathcal{B}\}$. We define the distortion function $f : R(h) \to R(g)$ by

$$f(h(A)) = g(A) \text{ for } A \in \mathcal{B}.$$

Then by the h-invariance, f is well-defined on $R(h)$. Since g is strongly h-invariant, f is non-decreasing on $R(h)$. We can extend f on $[0,\infty]$ by

$$f(t) = \inf\{f(h(A)) : t \leq h(A) \in R(h)\}.$$

This completes the proof.

DEFINITION 10. The fuzzy measure h is said to have the Darboux Property if for every $s, t \in R(h)$ with $s \leq t$, there exist $A, B \in \mathcal{B}$ with $A \subset B$ such that $t = h(A)$ and $s = h(B)$, where $R(h)$ is the range of h(see the Proof of THEOREM 1.

THEOREM 2(Honda and Okazaki[2]). Let g and h be two fuzzy measures on the σ-algebra $\mathcal{B}$. Suppose that h has the Darboux Property and that g is h-invariant. Then g is a distortion of h.

Proof. As in the Proof of THEOREM 1, we set $f : R(h) \to R(g)$ by

$$f(h(A)) = g(A) \text{ for } A \in \mathcal{B}.$$

Then f is well-defined by the h-invariance. Furthermore f is non-decreasing on $R(h)$ by the Darboux property. In fact, for every $s, t \in R(h)$ with $s \leq t$, there exist $A, B \in \mathcal{B}$ with $A \subset B$ such that $t = h(A)$ and $s = h(B)$. So we have $f(s) = f(h(A)) = g(A) \leq g(B) = f(h(B)) = f(t)$, by the monotonicity of g. Similarly to the Proof of THEOREM 1, we can extend f on $[0, \infty]$. This proves the THEOREM.

III. POSSIBILITY MEASURE

Let $\mathcal{B}$ be a σ-algebra on the set X and let μ be a σ-additive measure on $\mathcal{B}$. μ is called a probability in the case where $\mu(X) = 1$. μ is called a σ-finite measure if there exists a sequence $\{A_n\} \subset \mathcal{B}$ with $\mu(A_n) < \infty$ such that $X = \cup_{n=1}^{\infty} A_n$. In the sequel of this section, we assume that each singleton $\{x\} \in \mathcal{B}$.

Let $\Pi : \mathcal{B} \to [0, 1]$ be a possibility measure. We set the subset N, P and M as follows:

$$N = \{x \in X : \Pi(\{x\}) = 0\},$$

$$P = \{x \in X : 0 < \Pi(\{x\}) < 1\} \text{ and}$$

$$M = \{x \in X : \Pi(\{x\}) = 1\}.$$

If Π is a distortion of a σ-additive measure μ, then the distortion function f satisfies $f(0) = 0$ and that $f(\mu(X)) = \Pi(X) = 1$.

LEMMA 1. Suppose that the possibility measure Π is a distortion of a σ-finite σ-additive measure μ with the distortion function f. Then the set $P \cup M$ is at most countable.

Proof. For every $x \in P \cup M$, it holds that $0 < \Pi(\{x\}) = f(\mu(\{x\})) \leq 1$. It follows that $0 < \mu(\{x\}) \leq \mu(X)$. Since μ is σ-finite, $\mu(\{x\}) < \infty$. So we have $0 < \mu(\{x\}) < \infty$ for every $x \in P \cup M$. Since μ is σ-finite, each disjoint family of subsets of positive measures in $\mathcal{B}$ is at most countable, and so is $P \cup M$. This proves the LEMMA 1.

LEMMA 2. Suppose that the possibility measure Π is completely additive and is a distortion of a σ-additive measure μ with the distortion function f. Then the set P is at most countable.

Proof. Since Π is completely additive, it follows that $\Pi(A) = \sup\{\Pi(\{x\}) : x \in A\}$. We set for $0 < \epsilon < 1$, $P_\epsilon = \{x \in X : 0 < \Pi(\{x\}) \leq 1 - \epsilon\}$. Then by the complete additivity we have $\Pi(P_\epsilon) \leq 1 - \epsilon$, and hence $f(\mu(P_\epsilon)) \leq 1 - \epsilon$, which implies that $\mu(P_\epsilon) < \infty$. Consequently, P_ϵ is a countable set and $P = \cup_{n=1}^{\infty} P_{1/n}$ is at most countable. This proves the LEMMA 2.

LEMMA 3. Suppose that the possibility measure Π is a distortion of a probability measure μ with the distortion function f. Then it holds that either the set M is finite or $P = \phi$. In the case where $P = \phi$, Π is a $0 - 1$ fuzzy measure.

Proof. Suppose that M is not a finite set. Then there exists a sequence $\{x_n\} \subset M$. Since $1 = \Pi(\{x_n\}) = f(\mu(\{x_n\}))$, we have $\mu(\{x_n\}) > 0$. Since μ is a probability measure, it follows that $\mu(\{x_n\})$ converges to 0 as $n \to \infty$. So we have $1 = \lim_{n\to\infty} f(\mu(\{x_n\})) = f(0_+)$, where $f(0_+)$ is the right limit of $f(x)$ at $x = 0$(the right limit exists since f is non-decreasing). Consequently, it follows that $f(0) = 0$ and $f(x) = 1$ for $0 < x \leq 1$, which implies the assertion. This proves the LEMMA 3.

DEFINITION 11. Let $Z = \{z_n : n = 1, 2, ...\}$ be a countable set and let $f : Z \to [0, 1]$ be a function on Z. We say that Z is f-rearrangeable downwards if there exists an arrangement $Z = \{u_n : n = 1, 2, ...\}$ such that

$$f(u_1) \geq f(u_2) \geq f(u_3) \geq ....$$

DEFINITION 12. Let $Z = \{z_n : n = 1, 2, ...\}$ be a countable set and let $f : Z \to [0, 1]$ be a function on Z. We say that Z is f-rearrangeable upwards if there exists an arrangement $Z = \{v_n : n = 1, 2, ...\}$ such that

$$f(v_1) \leq f(v_2) \leq f(v_3) \leq ....$$

DEFINITION 13. Let $Z = \{z_n : n = 1, 2, ...\}$ be a countable set and let $f : Z \to [0, 1]$ be a function on Z. We say that Z is f-rearrangeable if there exists an arrangement $Z = \{..., w_{-3}, w_{-2}, w_{-1}, w_0, w_1, w_2, w_3, ...\}$ such that

$$..., f(w_{-3}) \leq f(w_{-2}) \leq f(w_{-1}) \leq f(w_0) \leq f(w_1) \leq f(w_2) \leq f(w_3) \leq ....$$

Let $\Pi : \mathcal{B} \to [0, 1]$ be a possibility measure. We set

$$\pi(x) = \Pi(\{x\}) \text{ for every } x \in X.$$

This function $\pi(x) : X \to [0, 1]$ is called the possibility distribution function of Π.

THEOREM 3-A. If the possibility measure Π is a distortion of a probability measure, then the set $P \cup M$ is countable and $P \cup M$ is $\pi(x)$-rearrangeable downwards.

Proof. Suppose that Π is a distortion of a probability measure μ with the distortion function f, that is, $\Pi(A) = f \circ \mu(A)$ for every $A \in \mathcal{B}$. Then by LEMMA 1, the set $P \cup M$ is countable. If the set $P \cup M$ is finite, then $P \cup M$ is $\pi(x)$-rearrangeable downwards. So we consider the case where $P \cup M$ is infinite. If $P = \phi$ then $P \cup M = M$ is $\pi(x)$-rearrangeable downwards by the any arrangement of M. If $P \neq \phi$, then the set M is finite (LEMMA 3) and hence P is infinite and is countable.

We claim that for every r with $1 \geq r > f(0_+)$, the set $Q_r = \{x \in P \cup M : \pi(x) \geq r\}$ is a finite set. In fact, if there exists an infinite sequence $\{x_n\} \subset Q_r$, it follows that $\pi(x_n) \geq r > f(0_+)$. On the other hand, similarly to the proof of LEMMA 3, we have $\mu(\{x_n\}) \to 0$ and $\pi(x_n) = f(\mu(\{x_n\}) \to f(0_+)$, which is a contradiction.

By the above claim, it follows that the set $\{x \in P \cup M : \pi(x) > f(0_+)\}$ is $\pi(x)$-rearrangeable downwards. In particular, if $\pi(x) > f(0_+)$ for every $x \in P \cup M$, then the set $P \cup M = \{x \in P \cup M : \pi(x) > f(0_+)\}$ is $\pi(x)$-rearrangeable downwards.

We shall investigate the last case where there is an $x \in P \cup M$ such that $\pi(x) = f(0_+)$(remark that $\pi(x) \geq f(0_+)$ for every $x \in P \cup M$, since $\mu(\{x\}) > 0$ for every $x \in P \cup M$. In this case, it holds that the set $\{x \in P \cup M : \pi(x) > f(0_+)\}$ is finite. To see this, suppose that the set $\{x \in P \cup M : \pi(x) > f(0_+)\}$ is infinite. Then we can take an infinite sequence $\{x_n : n = 1, 2, ...\} \subset P \cup M$ with $\pi(x_n) > f(0_+)$. Since $\pi(x_n) = f(\mu(\{x_n\})) > f(0_+) = \pi(x) = f(\mu(\{x\}))$ and f is non-decreasing, it follows that $\mu(\{x_n\}) > \mu(\{x\}) > 0$. Hence we have $\mu(\{x_n : n = 1, 2, ...\}) = \infty$, which contradicts to $\mu(X) = 1$. This completes the Proof.

If Π is completely additive, then the converse is also valid in THEOREM 3-A as follows.

THEOREM 3-B. Suppose that Π is completely additive. If the set $P \cup M$ is countable and if $P \cup M$ is $\pi(x)$-rearrangeable downwards, then the possibility measure Π is a distortion of a probability measure.

Proof. Let $P \cup M = \{u_n : n = 1, 2, ...\}$ be a rearrangement such that $\pi(u_n) \geq \pi(u_{n+1}) > 0$ for every n. By the complete additivity, if $\Pi(A) > 0$, it holds that $\Pi(A) = \Pi(A \cap (P \cup M))$. For $A, B \subset P \cup M$ with $\Pi(A) < \Pi(B)$, we have $\min\{n : u_n \in B\} < \min\{n : u_n \in A\}$ since $\pi(u_n) \geq \pi(u_{n+1})$ for every n. If we set $k = \min\{n : u_n \in B\}$, we have $\Pi(B) = \pi(u_k)$. We set the probability μ as follows:$\mu(\{u_n\}) = 2/3^n$ and $\mu(A) = 0$ for every $A \subset N$. If $\Pi(A) > 0$ and if $\Pi(A) < \Pi(B)$ then $u_k \in B$ and $A \subset \{u_{k+1}, u_{k+2}, ...\}$. So it follows that $\mu(B) \geq \mu(\{u_k\}) = 2/3^k > \mu(\{u_{k+1}, u_{k+2}, ...\}) = 1/3^k \geq \mu(A)$. Thus we have proved that the possibility Π is strongly μ-invariant. By THEOREM 1, Π is a distortion of μ. This completes the Proof.

THEOREM 4-A. If the possibility measure Π is a distortion of a σ-finite σ-additive measure, then the set $P \cup M$ is countable and $P \cup M$ is $\pi(x)$-rearrangeable.

Proof. Suppose that Π is a distortion of a σ-finite σ-additive measure μ with the distortion function f, that is, $\Pi(A) = f \circ \mu(A)$ for every $A \in \mathcal{B}$. By LEMMA 1, the set $P \cup M$ is countable.

If the set P is finite, then $P \cup M$ is clearly $\pi(x)$-rearrangeable upwards. So in the sequel, we shall suppose that the set P is infinite and is countable.

[Claim 1:For every infinite sequence $\{z_n\} \subset P$, the real sequence $\{\pi(z_n)\}$ has an accumulation point which is either $f(0_+)$ or $f(\infty_-) = \lim_{n\to\infty} f(n)$]

Take an arbitrary infinite sequence $\{z_n\} \subset P$. Let α be $\alpha = \inf\{\mu(\{z_n\}) : n = 1, 2, ...\}$. Then if α equals 0, there exists a subsequence $z_{n'}$ such that $\mu(\{z_{n'}\}) \to 0$. So we obtain $\pi(z_{n'}) = f(\mu(\{z_{n'}\})) \to f(0_+)$ since f is non-decreasing. In the case where $\alpha > 0$, it holds that $\mu(\{z_n\}) \geq \alpha > 0$ for every n, which implies that $\mu(\{z_1, z_2, ..., z_n\}) \to \infty$ as $n \to \infty$. It follows that $\Pi(\{z_1, z_2, ..., z_n\}) = \max\{\pi(z_1), \pi(z_2), ..., \pi(z_n)\} = f(\mu(\{z_1, z_2, ..., z_n\})) \uparrow f(\infty_-)$. In particular, for every $\delta > 0$, there exists z_k such that $\pi(z_k) > f(\infty_-)-\delta$. In fact, if $\pi(z_n) \leq f(\infty_-) - \delta$, then $\max\{\pi(z_1), \pi(z_2), ..., \pi(z_n)\} \leq f(\infty_-) - \delta$ for every n, which is a contradiction. Thus in this case, $f(\infty_-)$ is the accumulation point of $\{z_n\}$. This proves the Claim 1.

[Claim 2:For every α and β satisfying $f(0_+) < \alpha < \beta < f(\infty_-)$, the set $Q_{\alpha,\beta} = \{x \in P \cup M : \alpha \leq \pi(x) \leq \beta\}$ is finite]

If $Q_{\alpha,\beta}$ is infinite, then by the Claim 1, the real numbers $\{\pi(x) : x \in Q_{\alpha,\beta}\} \subset [\alpha, \beta]$ has an accumulation point which is either $f(0_+)$ or $f(\infty_-)$. This contradicts to $f(0_+) < \alpha < \beta < f(\infty_-)$.

[Claim 3: For every $x \in P \cup M$, it holds that $f(0_+) \leq \pi(x) \leq f(\infty_-)$.

Since $\pi(x) = f(\mu(\{x\}))$, $\mu(\{x\}) < \infty$ and $\pi(x) > 0$ for $x \in P \cap M$, it follows that $f(\mu(\{x\})) \leq f(\infty_-), \pi(x) = f(\mu(\{x\})) \geq f(0_+)$, since f is non-decreasing.

Now we devide the proof as follows.

(1)The case where $\{x \in P \cup M : f(0_+) < \pi(x) < f(\infty_-)\} = \phi$:

In this case, we have $P \cup M = \{x \in P \cup M : \pi(x) = f(0_+)\} \cap \{x \in P \cup M : \pi(x) = f(\infty_-)\}$, which is clearly $\pi(x)$-rearrangeable.

(2)The case where $f(0_+) < \pi(z) < f(\infty_-)$ for some $z \in P \cup M$.

By Claim 1, the set $\{x \in P \cup M : \pi(x) = \pi(z)\}$ is finite. We shall show that the set $\{x \in P \cup M : f(0_+) \leq \pi(x) < \pi(z)\}$ is $\pi(x)$-rearrangeable downwards((3),(4) and (5) below), and the set $\{x \in P \cup M : \pi(z) < \pi(x) \leq f(\infty_-)\}$ is $\pi(x)$-rearrangeable upwards((6) and (7) below).

(3)If the set $\{x \in P \cup M : f(0_+) < \pi(x) < \pi(z)\}$ is finite, then $\{x \in P \cup M : f(0_+) \leq \pi(x) < \pi(z)\}$ is clearly $\pi(x)$-rearrangeable downwards.

(4)If the set $\{x \in P \cup M : f(0_+) < \pi(x) < \pi(z)\}$ is infinite and if $f(0_+) = 0$, then by Claim 2, $\{x \in P \cup M : f(0_+) \leq \pi(x) < \pi(z)\}$ is clearly $\pi(x)$-rearrangeable downwards.

(5)If the set $\{x \in P \cup M : f(0_+) < \pi(x) < \pi(z)\}$ is infinite and if $f(0_+) > 0$, then the set $\{x \in P \cup M : \pi(x) = f(0_+)\}$ is empty. In fact, take an infinite sequence $x_n \in \{x \in P \cup M : f(0_+) < \pi(x) < \pi(z)\}$. Then by Claim 1, $\pi(x_n) \to f(0_+)$. Suppose that there exists w with $\pi(w) = f(0_+) > 0$. By $f(\mu(\{x_n\})) =$

$\pi(x_n) > f(0_+) = f(\mu(\{w\}))$, it follows that $\mu(\{x_n\}) > \mu(\{w\}) > 0$. Similarly Claim 2, we have $\mu(\{x_1, ..., x_n\}) \to \infty$ and $\max\{\pi(x_i) : i = 1, 2, ...n\} = \Pi(\{x_1, ..., x_n\}) = f(\mu(\{x_1, ..., x_n\})) \to f(\infty_-)$, which contradicts to $\pi(x_i) < \pi(z) < f(\infty_-)$. Consequently, $\{x \in P \cup M : f(0_+) \leq \pi(x) < \pi(z)\}$ is clearly $\pi(x)$-rearrangeable downwards.

(6)If the set $\{x \in P \cup M : \pi(z) < \pi(x) < f(\infty_-)\}$ is finite, then the set $\{x \in P \cup M : \pi(z) < \pi(x) \leq f(\infty_-)\}$ is clearly $\pi(x)$-rearrangeable upwards.

(7)Let the set $\{x \in P \cup M : \pi(z) < \pi(x) < f(\infty_-)\}$ be infinite. In this case, we shall show that the set $\{x \in P \cup M : \pi(x) = f(\infty_-)\}$ is empty. On the contrary, suppose that there exists $w \in \{x \in P \cup M : \pi(x) = f(\infty_-)\}$. Take any infinite sequence $x_n \in \{x \in P \cup M : \pi(z) < \pi(x) < f(\infty_-)\}$. Then we have $f(0_+) < \pi(z) < \pi(x_n) < \pi(w) = f(\infty_-)$, which implies that $0 < \mu(\{z\}) \leq \mu(\{x_n\}) \leq \mu(\{w\})$ since f is non-decreasing. By the inequality $f(\mu(\{x_1, x_2, ..., x_n\})) = \Pi(\{x_1, x_2, ..., x_n\}) = \max\{\pi(x_i) : i = 1, 2, ..., n\} < f(\mu(\{w\}))$, which implies that $\mu(\{x_1, x_2, ..., x_n\}) \leq \mu(\{w\})$. Since $\mu(\{x_1, x_2, ..., x_n\}) = \sum_{i=1}^{n} \mu(\{x_i\}) \geq n\mu(\{z\})$. Itfollows that $\mu(\{w\}) = \infty$, which is a contradiction since μ is σ-finite. This proves the assertion.

Theorem 4-b. Suppose that Π is completely additive. If the set $P \cup M$ is countable and if $P \cup M$ is $\pi(x)$-rearrangeable, then the possibility measure Π is a distortion of a σ-finite σ-additive measure.

Proof. Let $P \cup M = \{..., y_{-m}, ..., y_0, ..., y_n, ...\}$ be an arrangement such that $... \leq \pi(y_{-m}) \leq \pi(y_{-m+1}) \leq ... \leq \pi(y_0) ... \leq \pi(y_n) \leq \pi(y_{n+1}) \leq ....$ We set the σ-finite measure μ as follows:$\mu(\{y_{-i}\}) = 2/3^i (i = 1, 2, ...)$, $\mu(\{y_i\}) = 2^{i+1} (i = 0, 1, 2, ...)$ and $\mu(A) = 0$ for every $A \subset N$. Then by the manner same to Theorem 3-b, Π is a distortion of μ.

If we assume in advance the complete additivity, the possibility measure which is a distortion of a σ-additive measure is characterized as follows.

Theorem 5-a. Suppose that Π is completely additive. If Π is a distortion of a σ-additive measure, then it follows either 1. or 2. below:

1) $\sup\{\pi(x) : x \in P\} = 1$ and P is $\pi(x)$-rearrangeable, or
2) $\sup\{\pi(x) : x \in P\} < 1$ and P is $\pi(x)$-rearrangeable downwards.

Proof. Suppose that Π is a distortion of a σ-additive measure μ with the distortion function f, that is, $\Pi(A) = f \circ \mu(A)$ for every $A \in \mathcal{B}$. Remark that if $P = \phi$, then $\sup\{\pi(x) : x \in P\} = 0$.

[Claim 4:If $P \neq \phi$, then for every $x \in P$, it follows that $\pi(x) \in [f(0_+), f(\infty_-)]$]

For every $x \in P$,we have $0 < \pi(x) < 1$ and $\pi(x) = f(\mu(\{x\}))$. So it holds that $0 < \mu(\{x\}) < \infty$ and hence $f(0_+) \leq f(\mu(\{x\})) = \pi(x) \leq f(\infty_-)$, since f is non-decreasing.

[Claim 5:For every α and β satisfying $f(0_+) < \alpha < \beta < f(\infty_-)$, the set $Q_{\alpha,\beta} = \{x \in P : \alpha \leq \pi(x) \leq \beta\}$ is finite]

The claim 5 follows similarly to the proof of the Claim 2 in the proof of Theorem 4-A.

(1) Consider the case where $\sup\{\pi(x) : x \in P\} = 1$. By Claim 4, we have $f(\infty_-) = 1$. Now we shall fix an element $z \in P$. Then the set $R = \{x \in P : \pi(z) \leq \pi(x) < 1\}$ is $\pi(x)$-rearrangeable upwards by Claim 5. We set $S = \{x \in P : f(0_+) \leq \pi(x) < \pi(z)\} = P \backslash R$. We shall show that S is $\pi(x)$-rearrangeable downwards. If there exists $w \in P$ with $\pi(w) = f(0_+)$, then it follows that S is finite similarly to (5) in the proof of Theorem 4-A, hence S is $\pi(x)$-rearrangeable downwards. If $S = \{x \in P : f(0_+) < \pi(x) < \pi(z)\}$, then S is $\pi(x)$-rearrangeable downwards by Claim 5. Consequently, we have proved that the set P is $\pi(x)$-rearrangeable.

(2) Suppose that $\sup\{\pi(x) : x \in P\} = \gamma < 1$. If $P = \phi$, then there is nothing to prove (the empty set is rearrangeable with respect to any function). We shall suppose that $P \neq \phi$ in the sequel. Now we fix an element $z \in P$. Firstly, we show the set $T = \{x \in P : \pi(z) < \pi(x) \leq \gamma\} = \{x \in P : \pi(z) < \pi(x) \leq f(\infty_-)\}$ is finite and hence $\pi(x)$-rearrangeable upwards. Suppose that T is infinite, then we can take an infinite sequence $x_n \in T$ such that $\pi(x_n) \to \delta \in [\pi(z), \gamma]$. Since $0 < \pi(z) = f(\mu(z)) < f(\mu(x_n)) = \pi(x_n)$, it follows that $0 < \mu(z) < \mu(x_n)$. If we set $A = \{x_n : n = 1, 2, ...\}$, then it holds that $\mu(A) = \sum \mu(x_n) = \infty$, and $\Pi(A) = f(\mu(A)) = 1$. By the complete additivity of Π, it follows that $\Pi(A) = \sup\{\pi(x_n) : n = 1, 2, ...\} \leq \gamma < 1$, which is a contradiction. Hence T is finite. We put $U = \{x \in P : f(0_+) \leq \pi(x) < \pi(z)\}$. Then similarly to (1) above, U is $\pi(x)$-rearrangeable downwards. By Claim 5, the set $\{x \in P : \pi(x) = \pi(z)\}$ is finite. So the set $P = U \cup \{x \in P : \pi(x) = \pi(z)\} \cup T$ is $\pi(x)$-rearrangeable.

Theorem 5-b. Suppose that Π is completely additive. If one of the following conditions 1. or 2. is valid, then Π is a distortion of a σ-additive measure.

1) $\sup\{\pi(x) : x \in P\} = 1$ and P is $\pi(x)$-rearrangeable, or
2) $\sup\{\pi(x) : x \in P\} < 1$ and P is $\pi(x)$-rearrangeable downwards.

Proof. (1) Let $P = \{..., y_{-m}, ..., y_0, ..., y_n, ...\}$ be an arrangement such that $... \leq \pi(y_{-m}) \leq \pi(y_{-m+1}) \leq ... \leq \pi(y_0) \leq ... \leq \pi(y_n) \leq \pi(y_{n+1}) \leq ...$. We set the σ-additive measure μ as follows:

1) $\mu(y_{-m}) = 2/3^m, m = 1, 2, 3, ...,,$
2) $\mu(y_n) = 2^{n+1}, n = 0, 1, 2, ...,$
3) $\mu(y) = 0$ for $y \in N$,
4) $\mu(y) = \infty$ for $y \in M$.

Then similarly to Theorem 3-B, Π is a distortion of μ.

(2) Let $P = \{..., y_{-m}, ..., y_0, ..., y_N\}$ be an arrangement such that $... \leq \pi(y_{-m}) \leq \pi(y_{-m+1}) \leq ... \leq \pi(y_0) \leq ... \leq$

$\pi(y_N)$. We set the σ-addirive measure μ as above but only for n=0,1,2,...,N. Then Π is a distortion of μ.

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