Distortion of Fuzzy Measures

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Abstract—Let $g$ and $\mu$ be fuzzy measures on the $\sigma$-algebra $\mathcal{B}$. $g$ is said to be a distortion of $\mu$ if there is a non-decreasing function $f : [0, \infty] \to [0, \infty]$ satisfying that $g(E) = f(\mu(E))$, $E \in \mathcal{B}$. The general condition for the distortion between two fuzzy measures are discussed. The distortion can be characterized by the strong invariance. The fuzzy measure $g$ is called strongly $\mu$-invariant if $\mu(A) \leq \mu(B)$ implies that $g(A) \leq g(B)$ for every $A, B \in \mathcal{B}$. Then $g$ is a distortion of $\mu$ if and only if $g$ is in strongly $\mu$-invariant. We give a characterization of the possibility measure a distortion of a $\sigma$-additive measure.

I. INTRODUCTION AND PRELIMINARIES

We shall discuss the distortion property between fuzzy measures. The class of distorted fuzzy measure was introduced and investigated by Chateauneuf[1], Schmeidler[4] and Yaari[6], see also Pap[3] and Wang and Khiir[5].

Let $X$ be a set and $\mathcal{B}$ be a $\sigma$-algebra of subsets of $X$, that is, $\mathcal{B}$ satisfy the following conditions:
1) $\emptyset \in \mathcal{B}$,
2) if $E \in \mathcal{B}$, then $E^c \in \mathcal{B}$, and
3) if $E_n \in \mathcal{B} (n = 1, 2, \ldots)$, then $\bigcup_{n=1}^{\infty} E_n \in \mathcal{B}$.

Definition 1. The set function $g : \mathcal{B} \to [0, \infty]$ is called the fuzzy measure if the following conditions are satisfied:
1) $g(\emptyset) = 0$, and
2) for every $A, B \in \mathcal{B}$ with $A \subset B$, it follows that $g(A) \leq g(B)$ (the additivity).

Definition 2. The set function $\mu : \mathcal{B} \to [0, \infty]$ is called the $\sigma$-additive measure if the following conditions are satisfied:
1) $\mu(\emptyset) = 0$, and
2) for every pairwise disjoint family $E_n \in \mathcal{B} (n = 1, 2, \ldots)$, it follows that $\mu(\bigcup_{n=1}^{\infty} E_n) = \sum_{n=1}^{\infty} E_n$ (the additivity).

Definition 3. The function $f : [0, \infty] \to [0, \infty]$ is called the scaling function (or the distortion function) if the following conditions are satisfied:
1) $f(0) = 0$, $f(\infty) = \infty$, and
2) $f$ is non-decreasing.

Let $f$ be a distortion function and $h : \mathcal{B} \to [0, \infty]$ be a fuzzy measure. Then the set function $g(A) = f(h(A))$, $A \in \mathcal{B}$, is also a fuzzy measure. We denote this fuzzy measure $g$ by $g = f \circ h$.

Definition 4. Let $g$ and $h$ be two fuzzy measures on the $\sigma$-algebra $\mathcal{B}$. Then $g$ is said to be a distortion of $h$ if there exists a distortion function $f$ satisfying that $g = f \circ h$.

Definition 5. Let $g$ and $h$ be two fuzzy measures on the $\sigma$-algebra $\mathcal{B}$. Then $g$ is said to be $h$-invariant if $h(A) = h(B), A, B \in \mathcal{B}$, implies that $g(A) = g(B)$.

Definition 6. Let $g$ and $h$ be two fuzzy measures on the $\sigma$-algebra $\mathcal{B}$. Then $g$ is said to be strongly $h$-invariant if $h(A) \leq h(B), A, B \in \mathcal{B}$, implies that $g(A) \leq g(B)$.

Definition 7. The set function $\Pi : \mathcal{B} \to [0, 1]$ is called the possibility measure if it holds that $\Pi(A) + \Pi(B) = \Pi(A) \cap \Pi(B) = \max\{\Pi(A), \Pi(B)\}$ for every $A, B \in \mathcal{B}$.

Definition 8. The fuzzy measure $g$ is continuous from below if for every increasing sequence $A_n \uparrow A, A_n \in \mathcal{B}$, it follows that $g(A_n) \uparrow g(A)$.

Definition 9. The fuzzy measure $g$ is said to be completely additive if for every directed family $\mathcal{A} = \{A_n \mid n \in I\} \subset \mathcal{B}$ increasing to $A$ (that is, for each $A_n, A_\beta \in \mathcal{A}$, there exists $A_\alpha \in \mathcal{A}$ such that $A_n \cup A_\beta \subset A_\alpha$, and $\bigcup_{n \in I} A_n = A$), it follows that $g(A_n) \uparrow g(A)$.

II. INVARIANCE AND DISTORTION

It is clear that if $g = f \circ h$ for a distortion function $f$, then $g$ is strongly $h$-invariant, and also $h$-invariant. We shall consider the converse.

Theorem 1 (Honda and Okazaki[2]). Let $g$ and $h$ be two fuzzy measures on the $\sigma$-algebra $\mathcal{B}$. Then $g$ is a distortion of $h$ if and only if $g$ is strongly $h$-invariant.

Proof. The necessity is clear. We show the converse. We set the ranges $R(g), R(h)$ of $g, h$, respectively, by $R(g) = \{g(A) : A \in \mathcal{B}\}$ and $R(h) = \{h(A) : A \in \mathcal{B}\}$. We define the distortion function $f : R(h) \to R(g)$ by

$$f(h(A)) = g(A) \quad \text{for} \quad A \in \mathcal{B}.$$ 

Then by the $h$-invariance, $f$ is well-defined on $R(h)$. Since $g$ is strongly $h$-invariant, $f$ is non-decreasing on $R(h)$. We can extend $f$ on $[0, \infty]$ by

$$f(t) = \inf\{f(h(A)) : t \leq h(A) \in R(h)\}.$$
This completes the proof.

**Definition 10.** The fuzzy measure \( h \) is said to have the Darboux Property if for every \( s, t \in \mathcal{H}(h) \) with \( s \leq t \), there exist \( A, B \in \mathcal{B} \) with \( A \subseteq B \) such that \( t = h(A) \) and \( s = h(B) \), where \( \mathcal{H}(h) \) is the range of \( h \) (see the Proof of Theorem 1).

**Theorem 2 (Honda and Okazaki [2]).** Let \( g \) and \( h \) be two fuzzy measures on the \( \sigma \)-algebra \( \mathcal{B} \). Suppose that \( h \) has the Darboux Property and that \( g \) is \( h \)-invariant. Then \( g \) is a distortion of \( h \).

Proof. As in the Proof of Theorem 1, we set \( f : h \to f(g) \) by

\[
   f(h(A)) = g(A) \quad \text{for } A \in \mathcal{B}.
\]

Then \( f \) is well-defined by the \( h \)-invariance. Furthermore \( f \) is non-decreasing on \( h(A) \) by the Darboux property. In fact, for every \( s, t \in \mathcal{H}(h) \) with \( s \leq t \), there exist \( A, B \in \mathcal{B} \) with \( A \subseteq B \) such that \( t = h(A) \) and \( s = h(B) \). So we have \( f(s) = f(h(A)) = g(A) \leq g(B) = f(h(B)) = f(t) \), by the monotonicity of \( g \). Similarly to the Proof of Theorem 1, we can extend \( f \) on \([0, \infty)\). This proves the Theorem.

**III. Possibility measure**

Let \( \mathcal{B} \) be a \( \sigma \)-algebra on the set \( X \) and let \( \mu \) be a \( \sigma \)-additive measure on \( \mathcal{B} \). \( \mu \) is called a probability in the case where \( \mu(X) = 1 \). \( \mu \) is called a \( \sigma \)-finite measure if there exists a sequence \( \{A_n\} \subseteq \mathcal{B} \) with \( \mu(A_n) < \infty \) such that \( X = \bigcup_{n=1}^{\infty} A_n \). In the sequel of this section, we assume that \( \mu \) is a probability measure.

Let \( \Pi : \mathcal{B} \to [0, 1] \) be a possibility measure. We set the subset \( N, P, M \) and \( M \) as follows:

\[
N = \{x \in X : \Pi(x) = 0\},
\]
\[
P = \{x \in X : 0 < \Pi(x) < 1\} \quad \text{and}
\]
\[
M = \{x \in X : \Pi(x) = 1\}.
\]

If \( \Pi \) is a distortion of a \( \sigma \)-additive measure \( \mu \), then the distortion function \( f \) satisfies \( f(0) = 0 \) and that \( f(\mu(X)) = \Pi(X) = 1 \).

**Lemma 1.** Suppose that the possibility measure \( \Pi \) is a distortion of a \( \sigma \)-finite \( \sigma \)-additive measure \( \mu \) with the distortion function \( f \). Then the set \( P \cup M \) is at most countable.

Proof. For every \( x \in P \cup M \), it holds that \( 0 < \Pi(x) = f(\mu(x)) \leq 1 \). It follows that \( 0 < \mu(x) \leq \mu(X) \). Since \( \mu \) is \( \sigma \)-finite, \( \mu(x) < \infty \). So we have \( 0 < \mu(x) < \infty \) for every \( x \in P \cup M \). Since \( \mu \) is \( \sigma \)-finite, each disjoint family of subsets of positive measures in \( \mathcal{B} \) is at most countable, and so is \( P \cup M \). This proves the Lemma.

**Lemma 2.** Suppose that the possibility measure \( \Pi \) is completely additive and is a distortion of a \( \sigma \)-additive measure \( \mu \) with the distortion function \( f \). Then the set \( P \) is at most countable.

Proof. Since \( \Pi \) is completely additive, it follows that \( \Pi(A) = \sup \{\Pi(x) : x \in A\} \). We set for \( 0 < c < 1 \), \( P = \{x \in X : 0 < \Pi(x) \leq 1 - c\} \). Then by the complete additivity we have \( \Pi(P) \leq 1 - c \), and hence \( f(\mu(P)) \leq 1 - c \), which implies that \( \mu(P) < \infty \). Consequently, \( P \) is a countable set and \( P = \bigcup_{n=1}^{\infty} A_n \) is at most countable. This proves the Lemma 2.

**Lemma 3.** Suppose that the possibility measure \( \Pi \) is a distortion of a probability measure \( \mu \) with the distortion function \( f \). Then it holds that either the set \( M \) is finite or \( P = \emptyset \). In the case where \( P = \emptyset \), \( \Pi \) is a \( 0-1 \) fuzzy measure.

Proof. Suppose that \( M \) is not a finite set. Then there exists a sequence \( \{x_n\} \subseteq M \). Since \( 1 = \Pi(\{x_n\}) = f(\mu(\{x_n\})) \), we have \( \mu(\{x_n\}) > 0 \). Since \( \mu \) is a probability measure, it follows that \( \mu(\{x_n\}) \) converges to \( 0 \) as \( n \to \infty \). So we have \( 1 = \lim_{n \to \infty} f(\mu(\{x_n\})) = f(0_+), \) where \( f(0_+) \) is the right limit of \( f(x) \) at \( x = 0 \) (the right limit exists since \( f \) is non-decreasing). Consequently, it follows that \( f(0) = 0 \) and \( f(x) = 1 \) for \( 0 < x \leq 1 \), which implies the assertion. This proves the Lemma 3.

**Definition 11.** Let \( Z = \{x_n : n = 1, 2, \ldots\} \) be a countable set and let \( f : Z \to [0, 1] \) be a function on \( Z \). We say that \( Z \) is \( f \)-rearrangeable downwards if there exists an arrangement \( Z = \{z_n : n = 1, 2, \ldots\} \) such that

\[
f(z_1) \geq f(z_2) \geq f(z_3) \geq \ldots
\]

**Definition 12.** Let \( Z = \{x_n : n = 1, 2, \ldots\} \) be a countable set and let \( f : Z \to [0, 1] \) be a function on \( Z \). We say that \( Z \) is \( f \)-rearrangeable upwards if there exists an arrangement \( Z = \{v_n : n = 1, 2, \ldots\} \) such that

\[
f(v_1) \leq f(v_2) \leq f(v_3) \leq \ldots
\]

**Definition 13.** Let \( Z = \{x_n : n = 1, 2, \ldots\} \) be a countable set and let \( f : Z \to [0, 1] \) be a function on \( Z \). We say that \( Z \) is \( f \)-rearrangeable downwards if there exists an arrangement \( Z = \{w_{\infty-3}, w_{\infty-2}, w_{\infty-1}, w_0, w_1, w_2, w_3, \ldots\} \) such that

\[
..., f(w_{\infty-2}) \leq f(w_{\infty-1}) \leq f(w_0) \leq f(w_1) \leq f(w_2) \leq f(w_3) \leq \ldots
\]

Let \( \Pi : \mathcal{B} \to [0, 1] \) be a possibility measure. We set

\[
\pi(x) = \Pi(\{x\}) \quad \text{for every } x \in X.
\]

This function \( \pi(x) : X \to [0, 1] \) is called the possibility distribution function of \( \Pi \).

**Theorem 3-A.** If the possibility measure \( \Pi \) is a distortion of a probability measure, then the set \( P \cup M \) is countable and \( P \cup M \) is \( \pi(x) \)-rearrangeable downwards.

Proof. Suppose that \( \Pi \) is a distortion of a probability measure \( \mu \) with the distortion function \( f \), that is, \( \Pi(A) = f \circ \mu(A) \) for every \( A \in \mathcal{B} \). Then by Lemma 1, the set \( P \cup M \) is countable. If the set \( P \cup M \) is finite, then \( P \cup M \) is \( \pi(x) \)-rearrangeable downwards. So we consider the case where \( P \cup M \) is infinite. If \( P = \emptyset \) then \( P \cup M = M \) is \( \pi(x) \)-rearrangeable downwards by the any arrangement of \( M \). If \( P \neq \emptyset \), then the set \( M \) is finite (Lemma 3) and hence \( P \) is infinite and is countable.

We claim that for every \( r \) with \( 1 \geq r > f(0_+) \), the set \( Q_r = \{x \in P \cup M : \pi(x) \geq r\} \) is a finite set. In
fact, if there exists an infinite sequence \( \{x_n\} \subseteq Q_x \), it follows that \( \pi(x_n) \geq r > f(0)_+ \). On the other hand, similarly to the proof of Lemma 3, we have \( \mu(\{x_n\}) \to 0 \) and \( \pi(x_n) = f(\mu(\{x_n\})) \to f(0)_+ \), which is a contradiction.

By the above claim, it follows that the set \( \{x \in P \cup M : \pi(x) > f(0)_+\} \) is \( \pi(x) \)-rearrangeable downwards. In particular, if \( \pi(x) > f(0)_+ \) for every \( x \in P \cup M \), then the set \( P \cup M = \{x \in P \cup M : \pi(x) > f(0)_+\} \) is \( \pi(x) \)-rearrangeable downwards.

We shall investigate the last case where there is an \( x \in P \cup M \) such that \( \pi(x) = f(0)_+ \)(remark that \( \pi(x) \geq f(0)_+ \) for every \( x \in P \cup M \), since \( \mu(\{x\}) > 0 \) for every \( x \in P \cup M \). In this case, it holds that the set \( \{x \in P \cup M : \pi(x) > f(0)_+\} \) is finite. To see this, suppose that the set \( \{x \in P \cup M : \pi(x) > f(0)_+\} \) is infinite. Then we can take an infinite sequence \( \{x_n : n \in 1, 2, \ldots\} \subseteq P \cup M \) with \( \pi(x_n) > f(0)_+ \). Since \( \pi(x_n) = f(\mu(\{x_n\})) > f(0)_+ = \pi(x) = f(\mu(\{x\})) \) and \( f \) is non-decreasing, it follows that \( \mu(\{x_n\}) > \mu(\{x\}) > 0 \). Hence we have \( \pi(\{x_n : n \in 1, 2, \ldots\}) = \infty \), which contradicts to \( \mu(\{x\}) = 1 \). This completes the proof.

If \( P \) is completely additive, then the converse is also valid in Theorem 3-A as follows.

**Theorem 3-A.** Suppose that \( P \) is completely additive. If the set \( P \cup M \) is countable and \( P \cup M \) is \( \pi(x) \)-rearrangeable downwards, then the possibility measure \( II \) is a distortion of a probability measure.

**Proof.** Let \( P \cup M = \{u_n : n \in 1, 2, \ldots\} \) be a rearrangement such that \( \pi(u_n) \geq \pi(u_{n+1}) > 0 \) for every \( n \). By the complete additivity, if \( II(A) > 0 \), it holds that \( II(A) = II(A \cap (P \cup M)) \). For \( A, B \subseteq P \cup M \) with \( II(A) < II(B) \), we have \( \{n : u_n \in B\} < \{n : u_n \in A\} \) since \( \pi(u_n) = \pi(u_{n+1}) > 0 \). We set the probability \( \mu \) as follows \( \mu(\{u_n\}) = 2/3^n \) and \( \mu(A) = 0 \) for every \( A \subseteq \mathbb{N} \). If \( II(A) > 0 \) then \( \mu(A) = 0 \). We have the probability \( \mu \) is \( \mu \)-invariant. By Theorem 1, II is a distortion of \( \mu \). This completes the proof.

**Theorem 4-A.** If the possibility measure II is a distortion of a \( \sigma \)-finite \( \sigma \)-additive measure, then the set \( P \cup M \) is countable and \( P \cup M \) is \( \pi(x) \)-rearrangeable.

**Proof.** Suppose that II is a distortion of a \( \sigma \)-finite \( \sigma \)-additive measure \( \pi \) with the distortion function \( f \), that is, \( II(A) = f \circ \pi(A) \) for every \( A \subseteq \mathbb{B} \). By Lemma 1, the set \( P \cup M \) is countable.

If the set \( P \) is finite, then \( P \cup M \) is clearly \( \pi(x) \)-rearrangeable downwards. So in the sequel, we shall suppose that the set \( P \) is infinite and is countable.

**Claim 1:** For every infinite sequence \( \{z_n\} \subseteq P \), the real sequence \( \{\pi(z_n)\} \) has an accumulation point which is either \( f(0)_+ \) or \( f(\infty) = \lim_{n \to \infty} f(n) \).

Take an arbitrary infinite sequence \( \{z_n\} \subseteq P \). Let \( \alpha = \inf \{\pi(z_n) : n = 1, 2, \ldots\} \). Then if \( \alpha \) equals 0, there exists a subsequence \( z_{n_k} \) such that \( \pi(z_{n_k}) \to 0 \). So we obtain \( \pi(z_{n_k}) = f(\mu(\{z_{n_k}\})) \to f(0)_+ \) since \( f \) is non-decreasing. In the case where \( \alpha \to 0 \), it holds that \( \pi(z_{n_k}) \geq \alpha > 0 \) for every \( n \), which implies that \( \pi(z_{n_k}) = f(\mu(\{z_{n_k}\})) \to f(0)_+ \) as \( n \to \infty \). It follows that \( II(\{z_{n_k}, z_{n_k}, \ldots\}) = \max \{\pi(z_1), \pi(z_2), \ldots, \pi(z_{n_k})\} = f(\mu(\{z_{n_k}, z_{n_k}, \ldots\})) \) \( f(\infty) \). In particular, for every \( \delta > 0 \), there exists \( z_k \) such that \( \pi(z_k) > f(\infty) - \delta \). In fact, if \( \pi(z_k) \leq f(\infty) - \delta \), then \( \max \{\pi(z_1), \pi(z_2), \ldots, \pi(z_{n_k})\} \leq f(\infty) - \delta \) for every \( n \), which is a contradiction. Thus in this case, \( f(\infty) \) is the accumulation point of \( \{z_n\} \). This proves the Claim 1.

**Claim 2:** For every \( \alpha \) and \( \beta \) satisfying \( f(0)_+ < \alpha < \beta < f(\infty) \), the set \( Q_{\alpha, \beta} = \{x \in P \cup M : \alpha < \pi(x) < \beta\} \) is infinite.

If \( Q_{\alpha, \beta} \) is infinite, then by the Claim 1, the real numbers \( \pi(x) : x \in Q_{\alpha, \beta} \) has an accumulation point which is either \( f(0)_+ \) or \( f(\infty) \). This contradicts to \( f(0)_+ < \alpha < \beta < f(\infty) \).

**Claim 3:** For every \( x \in P \cup M \), it holds that \( f(0)_+ \leq \pi(x) \leq f(\infty) \).

Since \( \pi(x) = f(\mu(\{x\})) \), \( \mu(\{x\}) < \infty \) and \( \pi(x) > 0 \) for \( x \in P \cap M \), it follows that \( f(\mu(\{x\})) \leq f(\infty) \), \( \pi(x) = f(\mu(\{x\})) \geq f(0)_+ \), since \( f \) is non-decreasing.

Now we derive the proof as follows.

(1) The case where \( \{x \in P \cup M : f(0)_+ < \pi(x) < f(\infty)\} = \emptyset \).

In this case, we have \( P \cup M = \{x \in P \cup M : \pi(x) = f(0)_+\} \cap \{x \in P \cup M : \pi(x) = f(\infty)\} \), which is clearly \( \pi(x) \)-rearrangeable.

(2) The case where \( f(0)_+ < \pi(x) < f(\infty) \) for some \( x \in P \cup M \).

By Claim 1, the set \( \{x \in P \cup M : \pi(x) = f(0)_+\} \) is finite. We shall show that the set \( \{x \in P \cup M : f(0)_+ \leq \pi(x) < f(\infty)\} \) is \( \pi(x) \)-rearrangeable downwards (6) and (7) below), and the set \( \{x \in P \cup M : \pi(x) < f(\infty)\} \) is \( \pi(x) \)-rearrangeable upwards (6) and (7) below).

(3) If the set \( \{x \in P \cup M : f(0)_+ \leq \pi(x) < f(\infty)\} \) is finite, then \( \{x \in P \cup M : f(0)_+ \leq \pi(x) < f(\infty)\} \) is clearly \( \pi(x) \)-rearrangeable downwards.

(4) If the set \( \{x \in P \cup M : f(0)_+ < \pi(x) < f(\infty)\} \) is infinite and if \( f(0)_+ = 0 \), then by Claim 2, \( \{x \in P \cup M : f(0)_+ < \pi(x) < f(\infty)\} \) is clearly \( \pi(x) \)-rearrangeable downwards.

(5) If the set \( \{x \in P \cup M : f(0)_+ < \pi(x) < f(\infty)\} \) is infinite and if \( f(0)_+ > 0 \), then the set \( \{x \in P \cup M : \pi(x) = f(0)_+\} \) is empty. In fact, take an infinite sequence \( x_n \in \{x \in P \cup M : f(0)_+ < \pi(x) < f(\infty)\} \). Then by Claim 1, \( \pi(x_n) \to f(0)_+ \). Suppose that there exists \( w \in \pi(w) = f(0)_+ > 0 \). By \( f(\mu(\{x_n\})) = \).
\(\pi(x_i) > f(0_+) = f(\mu(\{x\}))\), it follows that \(\mu(\{x_n\}) > \mu(\{x\}) > 0\). Similarly Claim 2, we have \(\mu(\{x_1, \ldots, x_n\}) \to \infty\) and max \(\{\pi(x_i) : i = 1, 2, \ldots, n\} = \Pi(\{x_1, \ldots, x_n\}) = f(\mu(\{x_1, \ldots, x_n\})) \to f(\infty_+),\) which contradicts to \(\pi(x) < \pi(\infty)\). Consequently, \(\{x \in P \cup M : f(0_+) \leq \pi(x) < \pi(\infty)\} \) is clearly \(\pi(x)\)-rearrangeable downwards.

(6) If the set \(\{x \in P \cup M : \pi(x) < \pi(x) < f(\infty)\}\) is finite, then the set \(\{x \in P \cup M : \pi(x) < \pi(x) \leq f(\infty)\}\) is clearly \(\pi(x)\)-rearrangeable upwards.

(7) Let the set \(\{x \in P \cup M : \pi(x) < \pi(x) < f(\infty)\}\) be infinite. In this case, we shall show that the set \(\{x \in P \cup M : \pi(x) = f(\infty)\}\) is empty. On the contrary, suppose that there exists \(w \in \{x \in P \cup M : \pi(x) = f(\infty)\}\). Take any infinite sequence \(x_n \in \{x \in P \cup M : \pi(x) < \pi(x) < f(\infty)\}\). Then we have \(f(0_+) < \pi(x) < \pi(x) = f(\infty),\) which implies that \(0 < \mu(\{x\}) \leq \mu(\{x\}) \leq f(\infty)\), since \(f\) is non-decreasing. By the inequality \(f(\mu(\{x_1, x_2, \ldots, x_n\})) = \Pi(\{x_1, x_2, \ldots, x_n\}) = \max \{\pi(x_i) : i = 1, 2, \ldots, n\}\), the function \(\mu(\{x, x_1, x_2, \ldots, x_n\}) \leq \mu(\{w\})\). Since \(\mu(\{x_1, x_2, \ldots, x_n\}) = \sum_{i=1}^{n} \mu(\{x_i\}) \geq \mu(\{x\}),\) it follows that \(\mu(\{w\}) = \infty,\) which is a contradiction since \(\pi\) is \(\pi(x)\)-finite. This proves the assertion.

**Theorem 4-B.** Suppose that \(\Pi\) is completely additive. If the set \(P \cup M\) is countable and if \(P \cup M\) is \(\pi(x)\)-rearrangeable, then the possibility measure \(\Pi\) is a distortion of a \(\pi(x)\)-finite \(\pi(x)\)-additive measure.

Proof. Let \(P \cup M = \{x_1, x_2, \ldots, y_1, \ldots, y_n, \ldots\}\) be an arrangement such that \(\ldots \leq \pi(y_{m-1}) \leq \pi(y_m) \leq \pi(y_{m+1}) \leq \ldots \leq \pi(y_0) \leq \pi(y_{n+1}) \leq \ldots\). We set the \(\pi(x)\)-finite measure \(\mu\) as follows: \(\mu(\{x_i\}) = 2^{|\pi(x)|}^i (i = 1, 2, \ldots)\). We set the \(\Pi(\{x_i\}) = \mu(\{x\})\). If \(A = \{x : n = 1, 2, \ldots\}\), then it holds that \(\mu(A) = \sum \mu(x_n) = \infty,\) and \(\Pi(\{A\}) = f(\mu(A)) = 1\). By the complete additivity of \(\Pi,\) it follows that \(\Pi(\{A\}) = \sup \{\pi(x) : n = 1, 2, \ldots\} \leq \gamma,\) which is a contradiction. Hence \(\Pi\) is finite. We put \(U = \{x : f(0_+) \leq \pi(x) < \pi(\infty)\}\). Then similarly to (1) above, \(U\) is \(\pi(x)\)-rearrangeable downwards. By Claim 5, the set \(\{x \in P : \pi(x) = \pi(\infty)\}\) is finite. So the set \(P = U \cup \{x \in P : \pi(x) = \pi(\infty)\}\) is \(\pi(x)\)-rearrangeable.

**Theorem 5-B.** Suppose that \(\Pi\) is completely additive. If one of the following conditions 1. or 2. is valid, then \(\Pi\) is a distortion of a \(\pi(x)\)-additive measure.

1. \(\sup \{\pi(x) : x \in P\} = 1\) and \(P\) is \(\pi(x)\)-rearrangeable, or
2. \(\sup \{\pi(x) : x \in P\} < 1\) and \(P\) is \(\pi(x)\)-rearrangeable downwards.

Proof. (1) Let \(P = \{x_m, \ldots, x_n, \ldots\}\) be an arrangement such that \(\ldots \leq \pi(y_{m-1}) \leq \pi(y_m) \leq \pi(y_{m+1}) \leq \ldots \leq \pi(y_0) \leq \pi(y_{n+1}) \leq \ldots\). We set the \(\pi(x)\)-additive measure \(\mu\) as follows: \(1. \mu(\{y_m\}) = 2^{|\pi(y)|}^m, m = 1, 2, \ldots,\)

\(\mu(\{y_n\}) = 2^{|\pi(y)|}^{n+1}, n = 0, 1, 2, \ldots,\)

\(3. \mu(y) = 0\) for \(y \in N,\)

\(4. \mu(y) = \infty\) for \(y \in M.\)

Then similarly to Theorem 3-B, \(\Pi\) is a distortion of \(\mu.\)

(2) Let \(P = \{x_m, \ldots, x_n, \ldots\}\) be an arrangement such that \(\ldots \leq \pi(y_{m-1}) \leq \pi(y_m) \leq \pi(y_{m+1}) \leq \ldots \leq \pi(y_0) \leq \pi(y_{n+1}) \leq \ldots\)
\( \pi (y_0, y_1) \). We set the \( \sigma \)-additive measure \( \mu \) as above but only for \( n=0, 1, 2, \ldots, N \). Then II is a distortion of \( \mu \).

REFERENCES


