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CHARACTERIZATION OF 0-1 POSSIBILITY MEASURE ON TOPOLOGICAL SPACE

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1 Introduction

Let X be a set and let $\mathscr{A}$ be a family of subsets of X satisfying that
1. ϕ (the empty set) $\in \mathscr{A}$,
2. $\mathscr{A}$ contains the one-point set $\{x\}$ for every $x \in X$, and
3. $X \in \mathscr{A}$.

DEFINITION 1. The set function $g : \mathscr{A} \to [0,1]$ is called a fuzzy measure if the following conditions are satisfied:
1. $g(\phi) = 0$, $g(X) = 1$, and
2. for every $A, B \in \mathscr{A}$ with $A \subset B$, it follows that $g(A) \leq g(B)$ (the monotonicity).

DEFINITION 2. The set function $\Pi : \mathscr{A} \to [0,1]$ is called the possibility measure if there exists a function $\pi : X \to [0,1]$ satisfying that
1. $\sup_{x \in X} \pi(x) = 1$, and
2. $\Pi(A) = \sup_{x \in A} \pi(x)$ for every $A \in \mathscr{A}$.

DEFINITION 3. The set function $N : \mathscr{A} \to [0,1]$ is called the necessity measure if the dual measure N^d is a possibility measure, where the dual measure is given by $N^d(A) = 1 - N(A^c)$ for $A \in \mathscr{A}$.

DEFINITION 4. The fuzzy measure $g : \mathscr{A} \to [0,1]$ is called increasingly continuous if for every increasing $A_n \uparrow A$ with $A_n, A \in \mathscr{A}$, it holds that $g(A_n) \uparrow g(A)$.

DEFINITION 5. The fuzzy measure $g : \mathscr{A} \to [0,1]$ is called decreasingly continuous if for every decreasing $A_n \downarrow A$ with $A_n, A \in \mathscr{A}$, it holds that $g(A_n) \downarrow g(A)$.

The possibility measure Π is increasingly continuous and the necessity measure N is decreasingly continuous.

DEFINITION 6. The support S_g of the fuzzy measure g is defined by

$$S_g = \{x \in X \mid g(\{x\}) > 0\}.$$

Let $\Pi : \mathscr{A} \to [0,1]$ be a 0-1 possibility measure, that is, $\Pi(A) = 0$ *or* 1 for every $A \in \mathscr{A}$. In this case, the support of Π is given by $S_\Pi = \{x \in X \mid \Pi(\{x\}) = \pi(x) = 1\}$. If $\Pi(A) = \sup_{x \in A} \pi(x) = 1$, then there exists a sequence $x_n \in A$ such that $\lim \pi(x_n) = 1$. Since $\pi(x_n) = \Pi(\{x\}) = 0$ *or* 1, it follows that $\pi(x_k) = \Pi(x_k) = 1$ for some x_k. This means that $\Pi(A) = 0$ if and only if $A \cap S_\Pi = \phi$.

Let $N : \mathscr{A} \to [0,1]$ be a 0-1 necessity measure, that is, $N(A) = 0$ *or* 1 for every $A \in \mathscr{A}$. Then it follows that $N(A) = 1$ if and only if $N^d(A^c) = 0$.

Let X be a topological space with the topology $\mathscr{O}$ (the family of all open subsets of X). Denote by $\mathscr{B}$ the Borel σ-algebra on X, that is, $\mathscr{B}$ is the minimal σ-algebra containing the family $\mathscr{O}$ of all open subsets of X.

In Section 2, the closedness of the support of a possibility measure on $\mathscr{B}$ is characterized by the continuity.

In Section 3, we introduce the regularity and smoothness for a 0-1 fuzzy measure and give a characterization of the 0-1 possibility measure on $\mathscr{B}$.

For the basic notions of the fuzzy measure and topological space, we refer to [1] and [2].

2 Closedness of the support of 0-1 possiblity

In this section, we characterize the closedness of the support of the 0-1 possibility by the continuity.

THEOREM 1. Let X be a metric space and $\mathscr{B}$ be the Borel σ-algebra. Let Π be the possibility measure on $\mathscr{B}$. Then the support S_Π of Π is closed if and only if for every decreasing sequence $K_n \downarrow K$ of compact subsets, $\lim \Pi(K_n) = \Pi(K)$.

PROOF. Suppose that the support S_Π is closed. Let $K_n \downarrow K$ be a decreasing sequence of compact subsets. If $\Pi(K_n) = 1$ for every n, then $K_n \cap S_\Pi \neq \phi$. Since $K_n \cap S_\Pi$ is a decreasing sequence of compact subsets, by the Cantor's intersection theorem, $K \cap S_\Pi$ is not empty. So we have $\Pi(K) = 1$.

Conversely, suppose that for every decreasing sequence $K_n \downarrow K$ of compact subsets, it holds that $\lim \Pi(K_n) = \Pi(K)$. We show the closedness of S_Π. For every x in the closure of S_Π, we can find $x_n \in S_\Pi$ such that $x_n \to x$ in X. We set $K_n = \{x_i | i \geq n\} \cup \{x\}$. Then K_n is a converging sequence including the limit point, hence K_n is compact and $K_n \downarrow \{x\}$. Since $\Pi(K_n) = 1$, it follows that $\lim \Pi(K_n) = \Pi(\{x\}) = 1$, which shows $x \in S_\Pi$. Thus S_Π is closed.

DEFINITION 7. Let X be a topological space and U be an open subset of X. We say that U is a co-compact set if the complement U^c is a compact subset.

Let $N : \mathscr{B} \to [0,1]$ be a 0-1 necessity measure on the Borel σ-algebra $\mathscr{B}$. Then the dual measure N^d is a 0-1 possibility measure. Denote by S_{N^d} the support of N^d. Then, $N(A) = 1$ if and only if $N^d(A^c) = 0$, that is, $A^c \cap S_{N^d} = \phi$. Consequently, $N(A) = 1$ if and only if $A \supset S_{N^d}$.

THEOREM 2. Let X be a metric space and $\mathscr{B}$ be the Borel σ-algebra. Let N be the necessity measure on $\mathscr{B}$. Then the support S_{N^d} of the dual measure N^d is closed if and only if for every increasing sequence $U_n \uparrow U$ of co-compact subsets, $\lim N(U_n) = N(U)$.

PROOF. By Theorem 1, S_{N^d} is closed if and only if for every decreasing sequence $K_n \downarrow K$ of compact subsets, $\lim N^d(K_n) = N^d(K)$, that is $\lim N(K_n^c) = N(K^c)$. If we set $U_n = K_n^c$ and $U = K^c$, then each U_n is co-compact and $U_n \uparrow U$. This proves the Theorem 2.

3 Characterization of 0-1 possibility measure

DEFINITION 8. Let X be a topological space and $\mathscr{B}$ be the Borel σ-algebra on X. A fuzzy measure $g : \mathscr{B} \to [0,1]$ is called increasingly smooth if for every family $\{O_\lambda\}$ of open subsets with $g(O_\lambda) = 0$, it holds that $G(\bigcup_\lambda O_\lambda) = 0$. g is called decreasingly smooth if for every family $\{F_\lambda\}$ of closed subsets with $g(F_\lambda) = 1$, it holds that $g(\bigcap_\lambda F_\lambda) = 1$. We say g is smooth if g is increasingly smooth and decreasingly smooth.

The possibility measure on the Borel σ-algebra is increasingly smooth and the necessity measure is decreasingly smooth. We show the converse for 0-1 fuzzy measure.

THEOREM 3. Let X be a topological space and $\mathscr{B}$ be the Borel σ-algebra on X. Let g be a 0-1 fuzzy measure on $\mathscr{B}$. Suppose that g is outer regular, that is, for every $A \in \mathscr{B}$ with $g(A) = 0$ there exists an open subset O with $A \subset O$ such that $g(O) = 0$. If g is increasingly smooth, then g is a possibility measure.

PROOF. Let S_g be the support of g. Then for every $x \in S_g^c, g(\{x\}) = 0$, by the outer regularity, there exists an open subset O with $x \in O$ such that $g(O) = 0$. Consequently, the set $\{x \in X \mid g(\{x\}) = 0\} = S_g^c$ is open, and hence the support S_g is closed. Furthermore, by the increasingly smoothness, we have $g(S_g^c) = 0$. So $g(A) = 1$ if and only if $A \cap S_g \neq \phi$. This shows that g eqals the possibility measure with the support S_g.

THEOREM 4. Let X be a topological space and $\mathscr{B}$ be the Borel σ-algebra on X. Let g be a 0-1 fuzzy measure on $\mathscr{B}$. Suppose that g is inner regular, that is, for every $B \in \mathscr{B}$ with $g(B) = 1$ there exists an closed subset F with $B \supset F$ such that $g(F) = 1$. If g is deceasingly smooth, then g is a necessity measure.

PROOF. Let g^d be the dual measure of g. Then it is clear that g^d is outer regular and increasingly smooth by the definition of the dual measure. So by Theorem 3, g^d is a possibility measure and g is a necessity measure.

References

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