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Canonical fuzzy measure on (0, 1]

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Abstract

The fuzzy measure g on the unit interval $(0,1]$ is called canonical if g is a distortion of the Lebesgue measure. The characterization of the canonical fuzzy measure is given by the invariance properties against translations.

Keywords: Fuzzy measure; Canonical fuzzy measure; Distorted probability; Distortion function; Invariant against translations; Length invariant; Lebesgue measure

1. Introduction

In this paper, we consider the fuzzy measure g on the unit interval $(0,1]$ which is defined on the algebra $\mathscr{C}$ of all finite unions of the half-open intervals of the form $(a,b]$, $0<a<b\leqslant 1$. We are concerned with the fuzzy measure g which is a distortion of the Lebesgue measure μ.

We say that the fuzzy measure g on $\mathscr{C}$ is canonical if g is a distortion of the Lebesgue measure μ, that is, there exists a non-decreasing function $f:[0,1]\to[0,1]$ such that $f(0)=0$, $f(1)=1$ and $g(A)=f(\mu(A))$, $A\in\mathscr{C}$. In other words, the canonical fuzzy measure g originates in the Lebesgue measure μ through the non-decreasing distortion function f. Our canonicality is a special case of the distorted probability due to Chateauneuf [1], Schmeidler [5] and Yaari [7] (see also [4, 2.2.9]). As mentioned by Chateauneuf [1], it is crucial that the distortion function f is non-decreasing. In the case where f is continuous and strictly increasing, g is called a quasi-measure (see [6, 3.3]).

In this paper, we characterize the canonical fuzzy measure g on $(0,1]$ by the invariances against translations. Kruse [3] proved that if g_λ is a λ-additive fuzzy measure on $(0,1]$ which is invariant against translations, then g_λ can be represented uniquely as $g_\lambda(A)=f_\lambda(\mu(A))$, where $f_\lambda(x)=-1/\lambda+(1+\lambda)^x/\lambda$ and μ is the Lebesgue measure. We shall investigate the general conditions which assure that the fuzzy measure g is canonical.

2. Preliminaries

Let $\mathscr{A}$ be a class of subsets of the unit interval $(0,1]$ such that $\phi,(0,1]\in\mathscr{A}$. In this paper, to consider the Lebesgue measure on $\mathscr{A}$, we suppose that $\mathscr{A}$ is a subfamily of the Borel σ-algebra $\mathscr{B}$, where the Borel

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σ-algebra $\mathscr{B}$ is the σ-algebra generated by all open intervals in $(0,1]$.

Definition 1. The set function $g:\mathscr{A}\to[0,1]$ is called the fuzzy measure if the following conditions are satisfied:
1. $g(\phi)=0$, $g((0,1])=1$ and
2. for every $A,B\in\mathscr{A}$ with $A\subset B$, it follows that $g(A)\leqslant g(B)$ (the monotonicity).

Definition 2. The fuzzy measure $g:\mathscr{A}\to[0,1]$ is called continuous if for each monotone (increasing or decreasing) sequence $A_n\in\mathscr{A}$ satisfying $\lim_n A_n\in\mathscr{A}$, it holds that $\lim_n g(A_n)=g(\lim_n A_n)$.

Definition 3. Let λ be $\lambda>-1$. The fuzzy measure g_λ is called *λ-additive* if for each $A,B\in\mathscr{A}$ with $A\cap B=\emptyset$, $A\cup B\in\mathscr{A}$, it holds that $g_\lambda(A\cup B)=g_\lambda(A)+g_\lambda(B)+\lambda g_\lambda(A)g_\lambda(B)$.

Definition 4. The function $f:[0,1]\to[0,1]$ is called the distortion function if the following conditions are satisfied:
1. $f(0)=0$, $f(1)=1$ and
2. f is non-decreasing.

Let h be a fuzzy measure on $\mathscr{A}$ and f be a distortion function. Then the set function $g(A)=f(h(A))$, $A\in\mathscr{A}$, is also a fuzzy measure. g is called the distortion of h with the distortion function f [1,4, 2.2.9; 5,7]. If h is continuous and if f is a continuous distortion function, then the fuzzy measure g is continuous. We denote the fuzzy measure g by $g=f\circ h$. In the case where f is strictly increasing, then f is called a T-function (see [6, 3.3]).

Definition 5. The fuzzy measure g on $\mathscr{A}$ is called canonical if there exists a distortion function $f:[0,1]\to[0,1]$ satisfying that $g=f\circ\mu$, where μ is the Lebesgue measure.

3. Characterization of the canonical fuzzy measure on $(0,1]$

Let $\mathscr{C}$ be the family of subsets in $(0,1]$ of the form

$$(a_1,b_1]\cup(a_2,b_2]\cup\cdots\cup(a_n,b_n],$$

where $0\leqslant a_1\leqslant b_1\leqslant a_2\leqslant b_2\leqslant\cdots\leqslant a_n\leqslant b_n$, $n=1,2,\ldots$. That is, the class $\mathscr{C}$ is the set of all finite disjoint unions of left-open and right-closed subintervals in $(0,1]$. This family $\mathscr{C}$ forms an algebra (Halmos [2, Chapter 1, Section 4]), that is, the following conditions are satisfied:
1. if $E,F\in\mathscr{C}$, then $E\cup F\in\mathscr{C}$, and
2. if $E\in\mathscr{C}$, then $E^c\in\mathscr{C}$.

We introduce two invariances of the fuzzy measure g against translations which are also shared by the Lebesgue measure μ on $(0,1]$.

Definition 6. Let g be a fuzzy measure on $\mathscr{C}$. We say that g is weakly invariant against translations if for every subinterval $(a,b]\in\mathscr{C}$, it follows that

$$g((a,b])=g((a,b]-a)=g((0,b-a]).$$

Theorem 1. *Let g be a fuzzy measure on $\mathscr{C}$ which is weakly invariant against translations. Then there exists a unique distortion function $f:[0,1]\to[0,1]$ such that*

$$g((a,b])=f(\mu((a,b])),$$

where μ is the Lebesgue measure.

Proof. We set $f(x)=g((0,x])$. Then f is non-decreasing since g is monotone. Clearly, we have $f(0)=0$ and $f(1)=1$. By the definition of the weak invariance of g, it follows that $g((a,b])=g((0,b-a])=f(b-a)=f(\mu((a,b]))$. This proves the theorem. $\square$

Definition 7. Let g be a fuzzy measure on $\mathscr{C}$. We say that g is strongly invariant against translations if for every subinterval $A=(u,v]$ and every $B=(a_1,b_1]\cup(a_2,b_2]\cup\cdots\cup(a_n,b_n]\in\mathscr{C}$ satisfying $0\leqslant u\leqslant v\leqslant a_1\leqslant b_1\leqslant a_2\leqslant b_2\leqslant\cdots\leqslant a_n\leqslant b_n$, it holds that

$$g(A\cup B)=g(A\cup[B-(a_1-v)]).$$

Remark that the right endpoint of the subinterval A coincides with the left endpoint of the subset $B-(a_1-v)$ and $A\cap B=\emptyset$, $A\cap(B-(a_1-v))=\emptyset$.

Suppose that g is strongly invariant. Then we have

$$g((a_1, b_1] \cup (a_2, b_2])$$

$$= g((a_1, b_1] \cup (b_1, b_2 - (a_2 - b_1)])$$

$$= g((a_1, b_2 - a_2 + b_1])$$

$$= g((0, (b_1 - a_1) + (b_2 - a_2)]).$$

By the induction, it follows that

$$g((a_1, b_1] \cup (a_2, b_2] \cup \cdots \cup (a_n, b_n])$$

$$= g\left(\left(0, \sum_{i=1}^{n}(b_i - a_i)\right]\right).$$

Theorem 2. *Let g be a fuzzy measure on $\mathscr{C}$ which is strongly invariant against translations. Then there exists a unique distortion function $f:[0,1]\to[0,1]$ such that*

$$g(A) = f(\mu(A)), \quad A \in \mathscr{C}.$$

Proof. We set $f(x) = g((0, x])$ as in Theorem 1. Then we have for each $A = (a_1, b_1] \cup (a_2, b_2] \cup \cdots \cup (a_n, b_n] \in \mathscr{C}$,

$$g(A) = g\left(\left(0, \sum_{i=1}^{n}(b_i - a_i)\right]\right)$$

$$= f\left(\sum_{i=1}^{n}(b_i - a_i)\right) = f(\mu(A)).$$

This proves the theorem. □

Denote by $\mathscr{B}$ the Borel σ-algebra on $(0,1]$. $\mathscr{B}$ is the smallest monotone class containing $\mathscr{C}$ (see [2, Chapter 1, Section 6, Theorem B]).

Theorem 3. *Let g be a fuzzy measure on the Borel σ-algebra $\mathscr{B}$ on $(0,1]$. Suppose that*
1. *$g((0,x]) = g((0,x))$ for every $x \in (0,1]$,*
2. *g is continuous, and*
3. *g is strongly invariant against translations on $\mathscr{C}$.*

Then, there exists a unique continuous distortion function $f:[0,1]\to[0,1]$ such that

$$g(A) = f(\mu(A)), \quad A \in \mathscr{B}.$$

Proof. If we set $f(x) = g((0,x])$, then it follows that $g(A) = f(\mu(A))$ for each $A \in \mathscr{C}$ by Theorem 2. We show the continuity of f. For every $t_n \downarrow t$ in $(0,1]$, we have $\lim_n (0, t_n] = \bigcap_n (0, t_n] = (0, t]$. By the continuity of g, it follows that $g((0,t_n]) \downarrow g((0,t])$, which implies that $f(t_n) \downarrow f(t)$. This shows the right continuity of f. For every $s_n \uparrow s$ in $(0,1]$, we have $\lim_n (0, s_n] = \bigcup_n (0, s_n] = (0, s)$. By assumptions 1 and 2, we have $f(s_n) = g((0,s_n]) \uparrow g((0,s)) = g((0,s])$ $f(s)$, which shows the left continuity of f.

We set $\mathscr{D} = \{A \in \mathscr{B} \mid g(A) = f(\mu(A))\}$. Then $\mathscr{D}$ is a monotone class since g is continuous and f is a continuous function on $[0,1]$. Hence we have $\mathscr{D} = \mathscr{B}$ by Halmos [2, Chapter 1, Section 6, Theorem B]. This completes the proof. □

Definition 8. Let g be a fuzzy measure on the algebra $\mathscr{A}$ which contains the family $\{(0,x] \mid x \in (0,1]\}$. We say that g is length invariant if it holds that $g(A) = g((0, \mu(A)])$ for every $A \in \mathscr{A}$.

If $\mathscr{A} = \mathscr{C} =$ *the algebra of all finite unions of subintervals* $(a,b]$, then g is length invariant if and only if g is strongly invariant against translations.

Theorem 4. *Let g be a fuzzy measure on the algebra $\mathscr{A}$ as in Definition* 8. *Then g is canonical if and only if g is length invariant.*

Proof. If g is canonical, then we have $g(A) = f(\mu(A))$ for $A \in \mathscr{A}$. So it follows that $g(A) = f(\mu(A)) = f(\mu((0,\mu(A)])) = g((0,\mu(A)])$, since $\mu(A) = \mu((0,\mu(A)])$. Conversely, suppose that g is length invariant. If we set $f(x) = g((0,x])$, then it follows that $f(\mu(A)) = g((0,\mu(A)]) = g(A)$, which implies the canonicality of g. □

Example. Let g_0 be a fuzzy measure on $\mathscr{B}$ the Borel σ-algebra on $(0,1]$ given by $g_0(A) = 1$ if A is uncountable, and $g_0(A) = 0$ if A is atmost countable. Let $g(A) = (g_0(A) + \mu(A))/2$. Then, g is strongly invariant against translations on $\mathscr{C}$, and there exists a distortion function f satisfying $g(A) = f(\mu(A))$ for each $A \in \mathscr{C}$. g is not continuous since $g((0,1/n]) = (1 + 1/n)/2$ does not converge to 0. Furthermore, for every uncountable subset A with $\mu(A) = 0$, we have $g(A) = 1/2$ and hence $g(A) \neq f(\mu(A))$. Thus, g is not length invariant on the Borel σ-algebra $\mathscr{B}$.

References

[1] A. Chateauneuf, Decomposable capacities, distorted probabilities and concave capacities, Math. Social Sci. 31 (1996) 19–37.
[2] P.R. Halmos, Measure Theory, Van Nostrand Reinhold, New York, 1969.
[3] R. Kruse, A note on λ-additive fuzzy measures, Fuzzy Sets and Systems 8 (1982) 219–222.
[4] E. Pap, Null-Additive Set Functions, Kluwer Academic Publishers, Dordrecht, 1995.
[5] D. Schmeidler, Subjective probability and expected utility without additivity, Econometrica 57 (1989) 571–587.
[6] Z. Wang, G.J. Klir, Fuzzy Measure Theory, Plenum Press, New York, 1992.
[7] M.E. Yaari, The dual theory of choice under risk, Econometrica 55 (1987) 95–115.