CHARACTERIZATION OF 0-1 POSSIBILITY MEASURE 
ON TOPOLOGICAL SPACE

Aoi Honda and Yoshiaki Okazaki

1 Introduction

Let $X$ be a set and let $\mathcal{A}$ be a family of subsets of $X$ satisfying that
1. $\emptyset$ (the empty set) $\in \mathcal{A}$,
2. $\mathcal{A}$ contains the one-point set $\{x\}$ for every $x \in X$, and
3. $X \in \mathcal{A}$.

**Definition 1.** The set function $g : \mathcal{A} \to [0, 1]$ is called a fuzzy measure if the following conditions are satisfied:
1. $g(\emptyset) = 0$, $g(X) = 1$, and
2. for every $A, B \in \mathcal{A}$ with $A \subseteq B$, it follows that $g(A) \leq g(B)$ (the monotonicity).

**Definition 2.** The set function $\Pi : \mathcal{A} \to [0, 1]$ is called the possibility measure if there exists a function $\pi : X \to [0, 1]$ satisfying that
1. $\sup_{x \in X} \pi(x) = 1$, and
2. $\Pi(A) = \sup_{x \in A} \pi(x)$ for every $A \in \mathcal{A}$.

**Definition 3.** The set function $N : \mathcal{A} \to [0, 1]$ is called the necessity measure if the dual measure $N^d$ is a possibility measure, where the dual measure is given by $N^d(A) = 1 - \Pi(A)$ for $A \in \mathcal{A}$.

**Definition 4.** The fuzzy measure $g : \mathcal{A} \to [0, 1]$ is called increasingly continuous if for every increasing $A_n \uparrow A$ with $A_n, A \in \mathcal{A}$, it holds that $g(A_n) \uparrow g(A)$.

**Definition 5.** The fuzzy measure $g : \mathcal{A} \to [0, 1]$ is called decreasingly continuous if for every decreasing $A_n \downarrow A$ with $A_n, A \in \mathcal{A}$, it holds that $g(A_n) \downarrow g(A)$.

The possibility measure $\Pi$ is increasingly continuous and the necessity measure $N$ is decreasingly continuous.

**Definition 6.** The support $S_g$ of the fuzzy measure $g$ is defined by
$$S_g = \{x \in X \mid g(\{x\}) > 0\}.$$

Let $\Pi : \mathcal{A} \to [0, 1]$ be a 0-1 possibility measure, that is, $\Pi(A) = 0$ or 1 for every $A \in \mathcal{A}$. In this case, the support of $\Pi$ is given by $S_\Pi = \{x \in X \mid \Pi(\{x\}) = \pi(x) = 1\}$. If $\Pi(A) = \sup_{x \in A} \pi(x) = 1$, then there exists a sequence $x_n \in A$ such that $\lim \pi(x_n) = 1$. Since $\pi(x_n) = \Pi(\{x\}) = 0$ or 1, it follows that $\pi(x) = \Pi(x) = 1$ for some $x$. This means that $\Pi(A) = 0$ if and only if $A \cap S_\Pi = \emptyset$. 
Let $N : \mathcal{A} \to [0, 1]$ be a 0-1 necessity measure, that is, $N(A) = 0$ or 1 for every $A \in \mathcal{A}$. Then it follows that $N(A) = 1$ if and only if $N^d(A^c) = 0$.

Let $X$ be a topological space with the topology $\mathcal{C}$ (the family of all open subsets of $X$). Denote by $\mathcal{B}$ the Borel $\sigma$-algebra on $X$, that is, $\mathcal{B}$ is the minimal $\sigma$-algebra containing the family $\mathcal{C}$ of all open subsets of $X$.

In Section 2, the closedness of the support of a possibility measure on $\mathcal{B}$ is characterized by the continuity.

In Section 3, we introduce the regularity and smoothness for a 0-1 fuzzy measure and give a characterization of the 0-1 possibility measure on $\mathcal{B}$.

For the basic notions of the fuzzy measure and topological space, we refer to [1] and [2].

2 Closedness of the support of 0-1 possibility

In this section, we characterize the closedness of the support of the 0-1 possibility by the continuity.

**Theorem 1.** Let $X$ be a metric space and $\mathcal{B}$ be the Borel $\sigma$-algebra. Let $\Pi$ be the possibility measure on $\mathcal{B}$. Then the support $S_\Pi$ of $\Pi$ is closed if and only if for every decreasing sequence $K_n \downarrow K$ of compact subsets, $\lim \Pi(K_n) = \Pi(K)$.

**Proof.** Suppose that the support $S_\Pi$ is closed. Let $K_n \downarrow K$ be a decreasing sequence of compact subsets. If $\Pi(K_n) = 1$ for every $n$, then $K_n \cap S_\Pi \neq \emptyset$. Since $K_n \cap S_\Pi$ is a decreasing sequence of compact subsets, by the Cantor’s intersection theorem, $K \cap S_\Pi$ is not empty. So we have $\Pi(K) = 1$.

Conversely, suppose that for every decreasing sequence $K_n \downarrow K$ of compact subsets, it holds that $\lim \Pi(K_n) = \Pi(K)$. We show the closedness of $S_\Pi$. For every $x$ in the closure of $S_\Pi$, we can find $x_n \in S_\Pi$ such that $x_n \to x$ in $X$. We set $K_n = \{x_i | i \geq n\} \cup \{x\}$. Then $K_n$ is a converging sequence including the limit point, hence $K_n$ is compact and $K_n \downarrow \{x\}$. Since $\Pi(K_n) = 1$, it follows that $\lim \Pi(K_n) = \Pi(\{x\}) = 1$, which shows $x \in S_\Pi$. Thus $S_\Pi$ is closed.

**Definition 7.** Let $X$ be a topological space and $U$ be an open subset of $X$. We say that $U$ is a co-compact set if the complement $U^c$ is a compact subset.

Let $N : \mathcal{B} \to [0, 1]$ be a 0-1 necessity measure on the Borel $\sigma$-algebra $\mathcal{B}$. Then the dual measure $N^d$ is a 0-1 possibility measure. Denote by $S_{N^d}$ the support of $N^d$. Then, $N(A) = 1$ if and only if $N^d(A^c) = 0$, that is, $A^c \cap S_{N^d} = \emptyset$. Consequently, $N(A) = 1$ if and only if $A \supseteq S_{N^d}$.

**Theorem 2.** Let $X$ be a metric space and $\mathcal{B}$ be the Borel $\sigma$-algebra. Let $N$ be the necessity measure on $\mathcal{B}$. Then the support $S_{N^d}$ of the dual measure $N^d$ is closed if and only if for every increasing sequence $U_n \uparrow U$ of co-compact subsets, $\lim N(U_n) = N(U)$. 
Characterization of 0-1 Possibility Measure on Topological Space

Proof. By Theorem 1, \( S_{N^c} \) is closed if and only if for every decreasing sequence \( K_n \downarrow K \) of compact subsets, \( \lim N(K_n) = N(K^c) \), that is \( \lim N(K_n^c) = N(K^c) \). If we set \( U_n = K_n^c \) and \( U = K^c \), then each \( U_n \) is co-compact and \( U_n \uparrow U \). This proves the Theorem 2.

3 Characterization of 0-1 possibility measure

Definition 8. Let \( X \) be a topological space and \( \mathcal{B} \) be the Borel \( \sigma \)-algebra on \( X \). A fuzzy measure \( g : \mathcal{B} \rightarrow [0, 1] \) is called increasingly smooth if for every family \( \{O_i\} \) of open subsets with \( g(O_i) = 0 \), it holds that \( G(\bigcup_i O_i) = 0 \). \( g \) is called decreasingly smooth if for every family \( \{F_i\} \) of closed subsets with \( g(F_i) = 1 \), it holds that \( g(\bigcap_i F_i) = 1 \). We say \( g \) is smooth if \( g \) is increasingly smooth and decreasingly smooth.

The possibility measure on the Borel \( \sigma \)-algebra is increasingly smooth and the necessity measure is decreasingly smooth. We show the converse for 0-1 fuzzy measure.

Theorem 3. Let \( X \) be a topological space and \( \mathcal{B} \) be the Borel \( \sigma \)-algebra on \( X \). Let \( g \) be a 0-1 fuzzy measure on \( \mathcal{B} \). Suppose that \( g \) is outer regular, that is, for every \( A \in \mathcal{B} \) with \( g(A) = 0 \) there exists an open subset \( O \) with \( A \subset O \) such that \( g(O) = 0 \). If \( g \) is increasingly smooth, then \( g \) is a possibility measure.

Proof. Let \( S_g \) be the support of \( g \). Then for every \( x \in S_g^c, g(\{x\}) = 0 \), by the outer regularity, there exists an open subset \( O \) with \( x \in O \) such that \( g(O) = 0 \). Consequently, the set \( \{x \in X \mid g(\{x\}) = 0 \} = S_g^c \) is open, and hence the support \( S_g \) is closed. Furthermore, by the increasingly smoothness, we have \( g(S_g^c) = 0 \). So \( g(A) = 1 \) if and only if \( A \cap S_g \neq \emptyset \). This shows that \( g \) equals the possibility measure with the support \( S_g \).

Theorem 4. Let \( X \) be a topological space and \( \mathcal{B} \) be the Borel \( \sigma \)-algebra on \( X \). Let \( g \) be a 0-1 fuzzy measure on \( \mathcal{B} \). Suppose that \( g \) is inner regular, that is, for every \( B \in \mathcal{B} \) with \( g(B) = 1 \) there exists an closed subset \( F \) with \( B \supseteq F \) such that \( g(F) = 1 \). If \( g \) is decreasingly smooth, then \( g \) is a necessity measure.

Proof. Let \( g^d \) be the dual measure of \( g \). Then it is clear that \( g^d \) is outer regular and increasingly smooth by the definition of the dual measure. So by Theorem 3, \( g^d \) is a possibility measure and \( g \) is a necessity measure.

References


Department of Control Engineering and Science
Kyushu Institute of Technology
Kawazu, Fukuoka 820-8502, JAPAN