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INVARIANCE OF FUZZY MEASURE

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1. Introduction

A non-negative monotone set function defined on a σ-algebra $\mathscr{B}$ is called fuzzy measure. Let g and μ be fuzzy measures. We say g is μ-originated if there exists a non-decreasing function $f: [0, 1] \to [0, 1]$ such that $g(A) = f(\mu(A)) = (f \circ \mu)(A)$ for every $A \in \mathscr{B}$. We also say g is μ-invariant if $\mu(A) = \mu(B)$ implies $g(A) = g(B)$ for A, $B \in \mathscr{B}$. If g is μ-originated then g is μ-invariant. In this paper, we consider the converse. We show that if g is μ-invariant and if μ has the Darboux property, then g is μ-originated.

2. Preliminaries

Let X be a set and $\mathscr{B}$ be a σ-algebra on X, that is, $\mathscr{B}$ satisfies the following conditions:

1) $\phi, X \in \mathscr{B}$,

2) if $A \in \mathscr{B}$, then $A^c \in \mathscr{B}$, and

3) if $A_n \in \mathscr{B}$ $(n = 1, 2, \cdots)$, then $\bigcup_{n=1}^{\infty} A_n \in \mathscr{B}$.

DEFINITION 1. The set function $g : \mathscr{B} \to [0, 1]$ is called a fuzzy measure if g satisfies that

1) $g(\phi) = 0$, $g(X) = 1$ and that

2) $A \subset B$, A, $B \in \mathscr{B}$, implies that $g(A) \leqq g(B)$ (the monotonicity).

DEFINITION 2. The set function $g : \mathscr{B} \to [0, 1]$ is called a probability measure if g satisfies that

1) $g(\phi) = 0$, $g(X) = 1$ and that

2) $A \cap B = \phi$, A, $B \in \mathscr{B}$, implies that $g(A \cup B) = g(A) + g(B)$.

DEFINITION 3. Let g, μ be two fuzzy measures. We say that g is μ-invariant if $\mu(A) = \mu(B)$ $(A, B \in \mathscr{B})$ implies that $g(A) = g(B)$.

DEFINITION 4. Let g, μ be two fuzzy measures. We say that g is strongly μ-invariant if $\mu(A) \leqq \mu(B)$ $(A, B \in \mathscr{B})$ implies that $g(A) \leqq g(B)$.

DEFINITION 5. Let g, μ be two fuzzy measures. We say that g is μ-originated if there exists a non-decreasing function $f: [0, 1] \to [0, 1]$ such that $g(A) = f(\mu(A)) = (f \circ \mu)(A)$ for every $A \in \mathscr{B}$.

DEFINITION 6. We say that the fuzzy measure g satisfies the countable chain condition if the following two conditions are satisfied:

1) $A_n \uparrow A$ (A_n, $A \in \mathscr{B}$) implies that $g(A_n) \uparrow g(A)$, and

2) $A_n \downarrow A$ (A_n, $A \in \mathscr{B}$) implies that $g(A_n) \downarrow g(A)$.

It is clear that if the fuzzy measure g is μ-originated, then g is strongly μ-invariant. If g is strongly μ-invariant then g is also μ-invariant as the following Lemma shows.

LEMMA 1. Let g, μ be two fuzzy measures. If g is strongly μ-invariant then g is μ-invariant.

PROOF. Suppose that $\mu(A) = \mu(B)$, that is, $\mu(A) \leqq \mu(B)$ and $\mu(A) \geqq \mu(B)$. Then by the strong μ-invariance, it follows that $g(A) \leqq g(B)$ and $g(A) \geqq g(B)$, which shows $g(A) = g(B)$.

3. Invariance

Let g be a fuzzy measure defined on the σ-algebra $\mathscr{B}$. We set the range $R(g)$ of g as follows:

$$R(g) = \{g(A) \mid A \in \mathscr{B}\} \subset [0, 1].$$

Suppose that g is μ-invariant. Then we can define the function

$$f: R(\mu) \to R(g), \quad f(\mu(A)) = g(A).$$

In fact, if $t = \mu(A) = \mu(B) \in R(\mu)$, then by the μ-invariance of g, it follows that $g(A) = g(B)$. Thus it holds that $f(\mu(A)) = f(\mu(B)) = f(t)$, hence f is well-defined. To assure the strong μ-invariance, we must show the followings:

1) $f: R(\mu) \to R(g)$ is non-decreasing, and

2) $f: R(\mu) \to R(g)$ is extensible to a non-decreasing function on $[0, 1]$ into $[0, 1]$.

THEOREM 1. g is μ-originated if and only if g is strongly μ-invariant.

PROOF. Define $f: R(\mu) \to R(g)$, $f(\mu(A)) = g(A)$ as above. Since g is strongly μ-invariant, f is non-decreasing on $R(\mu)$. We shall extend f on $[0, 1]$ as follows: for each $t \in [0, 1]$ we set

$$f(t) = \inf \{ f(\mu(A)) \mid t \leqq \mu(A),\ A \in \mathscr{B} \}$$
$$= \inf \{ f(s) \mid t \leqq s,\ s \in R(\mu) \}.$$

Then f is non-decreasing and $g(A) = f(\mu(A))$ for every $A \in \mathscr{B}$. This completes the proof.

Now we consider a condition with which μ-invariance implies μ-originatedness.

DEFINITION 7. We say that the fuzzy measure μ has the Darboux property if for every $s, t \in R(\mu)$ with $s < t$, there exist $A, B \in \mathscr{B}$ satisfying $A \subset B$, $\mu(A) = s$ and $\mu(B) = t$.

THEOREM 2. Suppose that the fuzzy measure μ has the Darboux property. If the fuzzy measure g is μ-invariant, then g is also μ-originated.

PROOF. Let $f : R(\mu) \to R(g)$ be $f(\mu(A)) = g(A)$ for $A \in \mathscr{B}$. We show f is non-decreasing on $R(\mu)$. For every $s < t$, $s, t \in R(\mu)$, by the Darboux property, there exist A and B in $\mathscr{B}$ such that $A \subset B$, $s = \mu(A)$ and $t = \mu(B)$. Then we have $f(s) = g(A) \leqq g(B) = f(t)$, which implies the assertion. By the manner same to the proof of Theorem 1, f is extended to the non-decreasing function on [0, 1] and satisfies $g = f \circ \mu$. This completes the Proof.

In the preceding two theorems, we can not expect the continuity of the intertwining function f between g and μ. In the sequel of this section, we investigate the continuity of f.

DEFINITION 8. We say that the fuzzy measure μ has the strong Darboux property if μ satisfy the following two conditions:

1) for every $s < t$, $s, t \in R(\mu)$ and for every $A \in \mathscr{B}$ with $s = \mu(A)$, there exists $C \in \mathscr{B}$ such that $C \supset A$ with $t = \mu(C)$, and

2) for every $s < t$, $s, t \in R(\mu)$ and for every $B \in \mathscr{B}$ with $t = \mu(B)$, there exists $D \in \mathscr{B}$ such that $D \subset B$ with $s = \mu(D)$.

LEMMA 2. Suppose that μ has the strong Darboux property and that μ satisfies the countable chain condition. Then the range $R(\mu)$ is closed in [0, 1].

PROOF. Denote by $\overline{R(\mu)}$ the closure of $R(\mu)$ in [0, 1]. For every t in $\overline{R(\mu)}$, there exists a sequence $\{t_n\} \subset R(\mu)$ such that $t_n \to t$. Taking a subsequence, we can assume that $t_n \uparrow t$ (increasing) or $t_n \downarrow t$ (decreasing).

(1) the case $t_n \uparrow t$ (increasing): We put $t_n = \mu(A_n) \uparrow t$ for some A_n in $\mathscr{B}$. By the strong Darboux property, we can find B_n in $\mathscr{B}$ such that $B_1 \subset B_2 \subset \cdots \subset B_n \subset B_{n+1} \subset \cdots$ and $t_n = \mu(B_n)$. By the countable chain condition, it follows that $\mu(A_n) =$

$\mu(B_n) \uparrow \mu(\bigcup_{n=1}^{\infty} B_n) = t$, so $t \in R(\mu)$, which proves $\overline{R(\mu)} \subset R(\mu)$. Thus it holds that $\overline{R(\mu)} = R(\mu)$.

(2) the case $t_n \downarrow t$ (decreasing): Let $t_n = \mu(A_n) \downarrow t$ for some A_n in $\mathscr{B}$. By the strong Darboux property, we can find C_n in $\mathscr{B}$ such that $C_1 \supset C_2 \supset \cdots \supset C_n \supset C_{n+1} \supset \cdots$ and $t_n = \mu(C_n)$. By the countable chain condition, it follows that $\mu(A_n) = \mu(C_n) \downarrow \mu(\bigcap_{n=1}^{\infty} C_n) = t$, so $t \in R(\mu)$, which proves $\overline{R(\mu)} \subset R(\mu)$. Thus it holds that $R(\mu) = R(\mu)$. This completes the proof.

THEOREM 3. Suppose that g and μ are two fuzzy measures on the σ-algebra $\mathscr{B}$ which satisfy the countable chain conditions. Suppose also that μ has the strong Darboux property. If g is μ-invariant then g is μ-originated. Furthermore, we can find a continuous intertwining function $f: [0, 1] \to [0, 1]$ with $g(A) = f(\mu(A))$ for $A \in \mathscr{B}$.

PROOF. (1) Let $f: R(\mu) \to R(g)$ be $f(\mu(A)) = g(A)$, $A \in \mathscr{B}$. Then f is non-decreasing on $R(\mu)$ by Theorem 2 and $R(\mu)$ is closed by Lemma 2. We show that f is continuous on $R(\mu)$. Let $t_n = \mu(A_n) \to t = \mu(A)$ in $R(\mu)$. Taking a subsequence, we can suppose that $t_n \uparrow t$ or $t_n \downarrow t$. If $t_n \uparrow t$, then by the manner same to the proof of Lemma 2, there exists an increasing sequence $\{B_n\}$ in $\mathscr{B}$ such that $\mu(A_n) = \mu(B_n) = t_n$ and $\mu(A) = \mu(\bigcup_{n=1}^{\infty} B_n) = t$. By the countable chain condition for g, we have $g(B_n) \uparrow g(\bigcup_{n=1}^{\infty} B_n) = f(\mu(\bigcup_{n=1}^{\infty} B_n)) = f(t)$. So we have $f(t_n) = f(\mu(B_n)) = g(B_n) \uparrow f(t)$. In the case of $t_n \downarrow t$, we can show $f(t_n) \downarrow f(t)$ similarly, see the second part in the proof of Lemma 2. Hence f is continuous.

(2) We shall extend f to the whole interval $[0, 1]$ continuously and non-decreasingly as follows. Since the complementary set $U = [0, 1] \setminus R(\mu)$ is open, U is written by the countable disjoint union of open intervals as $U = \bigcup_{n=1}^{\infty} (a_n, b_n)$. The end points a_n and b_n are in the closed set $R(\mu)$. Since $a_n < b_n$ and f is non-decreasing on $R(\mu)$, it follows that $f(a_n) \leqq f(b_n)$. In the interval (a_n, b_n), we set f be the straight line connecting the points $(a_n, f(a_n))$ and $(b_n, f(b_n))$ in $[0, 1] \times [0, 1]$. since f is non-decreasing and continuous on the closed set $R(\mu)$, the extended fuction is also continuous and non-decreasing on $[0, 1]$. This completes the proof.

4. Darboux Property

Let μ be a probability measure on the σ-algebra $\mathscr{B}$ satisfying the countable chain condition, that is, μ is σ-aditive. A set E is called an atom if $\mu(E) > 0$ and if for every set A in $\mathscr{B}$ with $A \subset E$, we have either $\mu(A) = 0$ or $\mu(A) = \mu(E)$. μ is called non-atomic if there exists no atom in $\mathscr{B}$. It is well known that if μ is non-atomic, then the range $R(\mu)$ equals $[0, 1]$, and moreover it holds that for every E in $\mathscr{B}$, $\{\mu(A) \mid A \subset E, A \in \mathscr{B}\} = [0, \mu(E)]$, see Dinculeanu [1], §2, Proposition 7, Halmos [2], §41, Exercises (1)-(3). From this fact, it follows that a non-atomic σ-aditive probability measure satisfies the strong Darboux property (Definition 8). In the case where μ is a fuzzy measure, the notion of the null set is defined as follows.

DEFINITION 9. Let μ be a fuzzy measure on the σ-algebra $\mathscr{B}$. Then a subset N in $\mathscr{B}$ is called a μ-null set if $\mu(A \cup N) = \mu(A)$ for every A in $\mathscr{B}$, see Sugeno and Murofushi [3], 9.5.2.

Taking account of this definition, we shall define the notion of the atom for a fuzzy measure.

DEFINITION 10. Let μ be a fuzzy measure on the σ-algebra $\mathscr{B}$. A set E is called an atom if E is not a μ-null set and if there exists a st U in $\mathscr{B}$ disjoint from E such that for every $C \subset E$, $C \in \mathscr{B}$, it holds either $\mu(U \cup C) = \mu(U)$ or $\mu(U \cup C) = \mu(U \cup E)$.

DEFINITION 11. A fuzzy measure μ on the σ-algebra $\mathscr{B}$ is called non-atomic if there exists no atom in $\mathscr{B}$.

THEOREM 4. Let μ be a fuzzy measure on the σ-algebra $\mathscr{B}$. Suppose that μ is non-atomic and that μ satisfies the countable chain condition.

(1) For every E in $\mathscr{B}$ and every α with $\mu(E) \leqq \alpha \leqq 1$, there exists an element F in $\mathscr{B}$ such that $E \subset F$ and $\mu(F) = \alpha$.

(2) For every E in $\mathscr{B}$ and every α with $0 \leqq \alpha \leqq \mu(E)$, there exists an element G in $\mathscr{B}$ such that $G \subset E$ and $\mu(G) = \alpha$.

In particular, μ satisfies the strong Darboux property.

PROOF. (1) Our proof is very similar to that of Dincleanu [1], §2, Proposition 7. By induction, we shall find two sequences $\{F_n\}$ and $\{G_n\}$ in $\mathscr{B}$ having the following properties:

1) $E \subset G_0 \subset G_1 \subset \ldots \subset G_n \subset \ldots \subset F_n \subset \ldots \subset F_2 \subset F_1 \subset F_0 \subset X$,

2) if we put

$$u_n = \inf\{\mu(H) \mid G_{n-1} \subset H \subset F_{n-1},\ \mu(H) \geqq \alpha\},$$

$$v_n = \sup\{\mu(K) \mid G_{n-1} \subset K \subset F_n,\ \mu(K) \leqq \alpha\},$$

then the sequence $\{u_n\}$ is decreasing, the sequence $\{v_n\}$ is increasing and we have $v_n \leqq \alpha \leqq u_n$ for every n,

3) there exist two sequences $\varepsilon_n \to 0$ and $\eta_n \to 0$ such that

$$u_n \leqq \mu(F_n) < u_n + \varepsilon_n,\quad v_n - \eta_n < \mu(G_n) \leqq v_n.$$

We fix a sequence $r_n \downarrow 0$ in advance. We set

$$u_0 = \inf\{\mu(H) \mid E \subset H,\ \mu(H) \geqq \alpha\}.$$

Then $\alpha \leqq u_0 \leqq 1$ and for $\varepsilon_0 > 0$, there exists a set F_0 with

$$E \subset F_0 \quad \text{and} \quad u_0 \leqq \mu(F_0) < u_0 + \varepsilon_0.$$

We set

$$v_0 = \sup\{\mu(K) \mid E \subset K \subset F_0,\ \mu(K) \leqq \alpha\}.$$

Then $\mu(E) \leqq v_0 \leqq \alpha$ and for $\eta_0 > 0$, there exists a set G_0 with

$$E \subset G_0 \subset F_0 \text{ and } v_0 - \eta_0 < \mu(G_0) \leqq v_0.$$

Let u_1 be

$$u_1 = \inf\{\mu(H) \mid G_0 \subset H \subset F_0,\ \mu(H) \geqq \alpha\}.$$

Then $u_0 \leqq u_1 < u_0 + \varepsilon_0$. If we take small $\varepsilon_1 > 0$ as $\varepsilon_1 < r_1$ and as $u_1 + \varepsilon_1 < u_0 + \varepsilon_0$, then there exists a set F_1 with

$$G_0 \subset F_1 \subset F_0 \quad \text{and} \quad u_1 \leqq \mu(F_1) \leqq u_1 + \varepsilon_1.$$

Let v_1 be

$$v_1 = \sup\{\mu(K) \mid G_0 \subset K \subset F_1,\ \mu(K) \leqq \alpha\}.$$

Then $v_0 - \eta_0 < v_1 \leqq v_0$ since $E \subset G_0 \subset K \subset F_1 \subset F_0$. If we take small η_1 as $\eta_1 < r_1$ and as $v_0 - \eta_0 < v_1 - \eta_1$, then there exists a set G_1 such that

$$G_0 \subset G_1 \subset F_1 \quad \text{and} \quad v_1 - \eta_1 < \mu(G_1) \leqq v_1.$$

Suppose that the sets $F_1, F_2, \cdots, F_n$, $G_1, G_2, \cdots, G_n$ and the sequences $\varepsilon_1, \ldots, \varepsilon_n$, $\eta_1, \ldots, \eta_n$ are obtained satisfying 1), 2), and 3). We set

$$u_{n+1} = \inf\{\mu(H) \mid G_n \subset H \subset F_n,\ \mu(H) \geqq \alpha\}.$$

Then $u_n \leqq u_{n+1} < u_n + \varepsilon_n$. If we take small ε_{n+1} as $\varepsilon_{n+1} < r_{n+1}$ and as $u_{n+1} + \varepsilon_{n+1} < u_n + \varepsilon_n$, there exists a set F_{n+1} such that

$$G_n \subset F_{n+1} \subset F_n \quad \text{and} \quad u_{n+1} \leqq \mu(F_{n+1}) < u_{n+1} + \varepsilon_{n+1}.$$

Now we set

$$v_{n+1} = \sup\{\mu(K) \mid G_n \subset K \subset F_{n+1},\ \mu(K) \leqq \alpha\}.$$

Then $v_n - \eta_n < v_{n+1} \leqq v_n$. If we take small $\eta_{n+1} > 0$ as $\eta_{n+1} < r_{n+1}$ and as $v_n - \eta_n < v_{n+1} - \eta_{n+1}$, there exists a set G_{n+1} such that

$$G_n \subset G_{n+1} \subset F_{n+1} \quad \text{and} \quad v_{n+1} - \eta_{n+1} < \mu(G_{n+1}) \leqq v_{n+1}.$$

This completes the induction.

Let $\inf u_n = u$ and $\sup v_n = v$. Then we have $\mu(E) \leqq v \leqq \alpha \leqq u$. Put $F = \cap F_n$ and $G = \cup G_n$, then $G_n \subset G \subset F \subset F_n$. By the countable chain condition, it follows that $\mu(G) = \lim \mu(G_n) = v$ and $\mu(F) = \lim \mu(F_n) = u$. Suppose that $F \setminus G$ is not a μ-null set. Then for every C in $\mathscr{B}$ with $C \subset F \setminus G$, we have $G_n \subset G \subset G \cup C \subset F \subset F_n$ for every n. If $\mu(G \cup C) \leqq \alpha$, then by 2) and 3), $v_n - \eta_n < \mu(G \cup C) \leqq v_{n+1}$, which implies that $\mu(G \cup C) = v = \mu(G)$. If $\mu(G \cup C) \geqq \alpha$, then by 2) and 3), it follows that $u_{n+1} \leqq \mu(G \cup C) < u_n + \varepsilon_n$, which implies that $\mu(G \cup C) = u = \mu(F) = \mu(G \cup (F \setminus G))$. Thus the set $F \setminus G$ is an atom, which contradicts to our assumption. Hence the set $F \setminus G$ is a μ-null set. So we obtain $v = \mu(G) = \mu(F) = u = \alpha$. This proves (1).

(2) In this case, by the manner completely same to Dincleanu [1], §2, Proposition 7, we can find the sequences A_n and B_n in $\mathscr{B}$ satisfying the following conditions:

4) $A_0 \subset A_1 \subset \cdots \subset A_n \subset \cdots \subset B_n \subset \cdots \subset B_1 \subset B_0 \subset E$,

5) If we put

$$a_n = \sup\{\mu(A) \mid A_{n-1} \subset A \subset B_{n-1},\ \mu(A) \leqq \alpha\}$$

$$b_n = \inf\{\mu(B) \mid A_n \subset B \subset B_{n-1},\ \mu(B) \geqq \alpha\}$$

the sequence $\{a_n\}$ is decreasing, $\{b_n\}$ is increasing and

$$a_n \leqq \alpha \leqq b_n \quad \text{for every } n,$$

6) there exist two sequences $\varepsilon_n \to 0$ and $\eta_n \to 0$ such that

$$a_n - \varepsilon_n < \mu(A_n) \leqq a_n \quad \text{and} \quad b_n \leqq \mu(B_n) < b_n + \eta_n.$$

Let $A = \cup A_n$ and $B = \cap B_n$, then by a slight modification of the first part of this proof, we can show the set $B \setminus A$ is a μ-null set, which implies $\mu(A) = \mu(B) = \alpha = \inf a_n = \sup b_n$.

5. Invariance between propability Measures

THEOREM 5. Let g and μ be σ-additive probability measures on the σ-algebra $\mathscr{B}$. If μ is non-atomic and if g is μ-invariant, then $g = \mu$.

PROOF. The range $R(\mu)$ equals the interval [0, 1] and there exists a continous function f with $g(A) = f(\mu(A))$ for A in $\mathscr{B}$. For every $s, t \geqq 0$ with $s + t \leqq 1$, take A, B such that $A \cap B = \phi$, $t = \mu(A)$ and $s = \mu(B)$. Then we have $f(s + t) = f(\mu(A \cup B)) = g(A \cup B) = g(A) + g(B) = f(t) + f(s)$, that is, f is a linear function. Since $f(1) = 1$, we have $f(t) = t$, so $g(A) = \mu(A)$ for A in $\mathscr{B}$.

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