GENERALIZATIONS OF THE Hlawka's Inequality

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This paper is dedicated to the late Professor Shigeru Itoh

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1. Introduction

We shall extend the Hlawka's 3-element inequality in $\mathbb{R}^n$ (or in Hilbertian space) to the $n$-element inequality in $L^1$ related to the Hanner's inequality and to the $n$-element inequality of another type, which is related closely to the Adamović inequality.

The original Hlawka's inequality is as follows (the 3-element Hlawka's inequality):

$$
\|x + y + z\| + \|x\| + \|y\| + \|z\| \geq \|x + y\| + \|y + z\| + \|x + z\| \quad \text{for } x, y, z \in \mathbb{R}^n.
$$

If we set by $x_1 = (x + y)/2$, $x_2 = (y + z)/2$, $x_3 = (z + x)/2$, then we have the following inequality equivalent to the Hlawka's inequality:

$$
\|x_1 + x_2 + x_3\| + \|x_1 + x_2 - x_3\| + \|x_1 - x_2 + x_3\| + \|x_1 + x_2 + x_3\|
\geq 2(\|x_1\| + \|x_2\| + \|x_3\|).
$$

This inequality is rewritten as $E\left\| \sum_{i=1}^3 \varepsilon_i x_i \right\| \geq \frac{1}{2} \sum_{i=1}^3 \|x_i\|$, where $\varepsilon_i$ is the Rademacher sequence ($\varepsilon_i = \pm 1$ with probability $\frac{1}{2}$) and $E$ means the expectation. We shall give the following extension:

$$
E\left\| \sum_{i=1}^n \varepsilon_i x_i \right\| \geq \frac{1}{2^{n-1}} \cdot \frac{n-1}{C_{[n]}^1} \cdot \sum_{i=1}^n \|x_i\| \quad \text{for } x_1, \ldots, x_n \in L^1.
$$

The constant $\frac{1}{2^{n-1}} \cdot \frac{n-1}{C_{[n]}^1}$ is best possible.

In the Euclidean space $\mathbb{R}^n$, the Hlawka's inequality is generalized by Adamović as follows:

$$
\left\| \sum_{i=1}^n x_i \right\| + (n - 2) \sum_{i=1}^n \|x_i\| \geq \sum_{1 \leq i < j \leq n} \|x_i + x_j\| \quad \text{for } x_1, \ldots, x_n \in \mathbb{R}^n.
$$

We shall prove that if the 3-element Hlawka's inequality is valid in the Banach space $E$, then it follows also that $\left\| \sum_{i=1}^n x_i \right\| + (n - 2) \sum_{i=1}^n \|x_i\| \geq \sum_{1 \leq i < j \leq n} \|x_i + x_j\|$ for $x_1, \ldots, x_n \in E$.

In particular this $n$-element inequality is valid in $L^1$. 

2. The Hlawka’s inequality and the 3-element Hanner’s inequality

Hlawka (see [3]) proved the following inequality: For \( x, y, z \in \mathbb{R}^n \), it holds that
\[
\|x + y + z\| + \|x\| + \|y\| + \|z\| \geq \|x + y\| + \|y + z\| + \|z + x\|.
\]
This inequality follows immediately from the triangular inequality and the following identity ([3]):
\[
\begin{align*}
(\|x + y + z\| + \|x\| + \|y\| + \|z\| - \|x + y\| - \|y + z\| - \|z + x\|) \\
\times (\|x + y + z\| + \|x\| + \|y\| + \|z\|) \\
= (\|y\| + \|z\| - \|y + z\|)(\|x\| - \|y + z\| + \|x + y + z\|) \\
+ (\|z\| + \|x\| - \|z + x\|)(\|y\| - \|z + x\| + \|x + y + z\|) \\
+ (\|x\| + \|y\| - \|x + y\|)(\|z\| - \|x + y\| + \|x + y + z\|) \\
\geq 0.
\end{align*}
\]
Let \((S, \Sigma, \mu)\) be a measure space. The norm of \(L^1\) is given by \(\|x\| = \int |x(t)|\,d\mu(t)\). The \(n\)-element Hanner’s inequality was obtained in [1], [2]. In the case of \(L^1\), the \(n\)-element Hanner’s inequality is as follows. Let \(n\) be a natural number, \(\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n\) be the independent Rademacher sequence and \(x_1, x_2, \ldots, x_n \in L^1\). Then it holds that
\[
E\left|\sum_{i=1}^{n} \varepsilon_i x_i\right| \geq E\sum_{i=1}^{n} |\varepsilon_i| \|x_i\|.
\]
In the case where \(n = 3\), by the triangular inequality, the Hanner’s inequality implies that
\[
\|x_1 + x_2 + x_3\| + \|x_1 + x_2 - x_3\| + \|x_1 - x_2 + x_3\| + \|-x_1 + x_2 + x_3\| \\
\geq \|x_1\| + \|x_2\| + \|x_3\| + \|x_1 - x_2 + x_3\| + \|x_1 - x_2 + x_3\| - \|x_1\| - \|x_2\| + \|x_3\| \\
+ \|x_1 + x_2 - x_3\| + \|x_1 - x_2 + x_3\| + \|-x_1 + x_2 + x_3\| - \|x_1 - x_2 + x_3\| + \|x_1\| + \|x_2\| + \|x_3\| \\
= 2(\|x_1\| + \|x_2\| + \|x_3\|).
\]
If we set by \(x = x_1 + x_2 - x_3,\ y = x_1 - x_2 + x_3,\ z = -x_1 + x_2 + x_3\), then it follows that
\[
\|x + y + z\| + \|x\| + \|y\| + \|z\| \geq \|x + y\| + \|y + z\| + \|z + x\|.
\]
Thus the Hlawka’s inequality is derived from the 3-element Hanner’s inequality. Hence the Hlawka’s inequality holds in \(L^1\). Conversely from the Hlawka’s inequality, we
obtain the 3-element Hanner's inequality in the following way. First, we have
\[
\|x + y + z\| + \|-x + y + z\| + \|x - y + z\| + \|x + y - z\|
\geq \|(x + y + z) + (x - y - z)\| + \|(x - y + z) + (x + y - z)\|
= \|2x\| + \|2x\| = 4\|x\|.
\]

Next, let \( u = -x + y + z, \ v = x - y + z, \ w = x + y - z \) and we apply the Hlawka’s inequality as follows;
\[
\|x + y + z\| + \|-x + y + z\| + \|x - y + z\| + \|x + y - z\|
= \|u + v + w\| + \|u\| + \|v\| + \|w\|
\geq \|u + v\| + \|v + w\| + \|w + u\|
= \|2z\| + \|2x\| + \|2y\| = 2(\|x\| + \|y\| + \|z\|).
\]

Hence, it holds that
\[
\|x + y + z\| + \|-x + y + z\| + \|x - y + z\| + \|x + y - z\|
\geq \left\{ \begin{array}{l}
4\|x\| \\
2(\|x\| + \|y\| + \|z\|)
\end{array} \right. \quad (*)
\]

We show the 3-element Hanner’s inequality:
\[
\|x_1 + x_2 + x_3\| + \|x_1 + x_2 - x_3\| + \|x_1 - x_2 + x_3\| + \|-x_1 + x_2 + x_3\|
\geq \|x_1\| + \|x_2\| + \|x_3\| + \|x_1\| + \|x_2\| - \|x_3\| + \|x_1\| - \|x_2\| + \|x_3\|
+ \|-x_1\| + \|x_2\| + \|x_3\|.
\]

We can suppose that \( \|x_1\| \geq \|x_2\| \geq \|x_3\| \) without loss of generality. Then the last term is:
\[
(\|x_1\| + \|x_2\| + \|x_3\|) + (\|x_1\| + \|x_2\| - \|x_3\|) + (\|x_1\| - \|x_2\| + \|x_3\|)
+ \|-x_1\| + \|x_2\| + \|x_3\|
= 3\|x_1\| + \|x_2\| + \|x_3\| + \|-x_1\| + \|x_2\| + \|x_3\|.
\]

Consider the two cases:

(i) if \( \|x_1\| \geq \|x_2\| + \|x_3\| \), then

right-hand side = 3\|x_1\| + \|x_2\| + \|x_3\| + \|x_1\| - \|x_2\| - \|x_3\| = 4\|x_1\|
(ii) If \( \|x_1\| \leq \|x_2\| + \|x_3\| \), then
\[
\text{right-hand side} = 3\|x_1\| + \|x_2\| + \|x_3\| - \|x_1\| + \|x_2\| + \|x_3\|
= 2(\|x_1\| + \|x_2\| + \|x_3\|).
\]
In these two cases, the Hanner's 3-element inequality is derived by (*).
Accordingly, the Hlawka's inequality and the 3-element Hanner's inequality are equivalent in \( L^1 \).

3. Generalization of Hlawka's inequality

The Hlawka's inequality in \( L^1 \) is given by
\[
E \left\| \sum_{i=1}^{3} \varepsilon_i x_i \right\| \geq \frac{1}{2} \sum_{i=1}^{3} \|x_i\| \quad \text{for} \quad x_1, x_2, x_3 \in L^1,
\]
where \( E \) means the expectation with respect to the Rademacher distribution. We shall extend the Hlawka's inequality naturally as follows. Let \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \) be the independent Rademacher sequence and \( x_1, x_2, \ldots, x_n \in L^1 \). We can conjecture the following inequality;
\[
E \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| \geq K(n) \cdot \sum_{i=1}^{n} \|x_i\|,
\]
where \( K(n) \) is a constant which depends on \( n \).

**Theorem 1.** Let \( n \) be a natural number, \( \varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n \) be the independent Rademacher sequence and \( x_1, x_2, \ldots, x_n \) be functions in \( L^1 \), then it holds that
\[
E \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| \geq \frac{1}{2^{n-1} \cdot n!} \cdot \sum_{i=1}^{n} \|x_i\|,
\]
where the constant \( \frac{1}{2^{n-1} \cdot n!} \) is best possible.

**Proof.** We shall start from the \( n \)-element Hanner's inequality. We have
\[
E \left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| \geq \left| \sum_{i=1}^{n} \varepsilon_i \|x_i\| \right|
= \frac{1}{2^n} \cdot \sum_{\text{sum for all choices of } \pm \text{ signs}} \left| \pm \|x_1\| \pm \|x_2\| \pm \cdots \pm \|x_n\| \right|
\]
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\[
= \frac{1}{2^n} \left| \| x_1 \| + \| x_2 \| + \cdots + \| x_n \| \right|
\]

\[
+ \sum \left( \text{sum for all choices of one minus signs} \right) \left| \| x_1 \| + \cdots - \| x_j \| + \cdots + \| x_n \| \right|
\]

\[
+ \sum \left( \text{sum for all choices of two minus signs} \right) \left| \| x_1 \| + \cdots - \| x_j \| + \cdots - \| x_n \| \right|
\]

\[
+ \cdots + \left| -\| x_1 \| - \| x_2 \| - \cdots - \| x_n \| \right|.
\]

And we use the triangular inequality \( \sum |(\cdots)| \geq | \sum (\cdots) | \). Among the terms

\[
\left( \sum_{\text{\small k minus signs}} \pm \| x_1 \| \pm \cdots \pm \| x_j \| \pm \cdots \pm \| x_n \| \right)
\]

with \( k \) minus signs, there are \( \binom{n-1}{k} \) terms in which the coefficient of \( \| x_j \| \) is +1, and there are \( \binom{n-1}{k-1} \) terms in which the coefficient of \( \| x_j \| \) is −1. If \( \binom{n-1}{k} - \binom{n-1}{k-1} \geq 0 \), then it holds that

\[
\sum \left( \text{sum for all choices of k minus signs} \right) (\cdots) = (\binom{n-1}{k} - \binom{n-1}{k-1}) \times (\| x_1 \| + \| x_2 \| + \cdots + \| x_n \|).
\]

If \( \binom{n-1}{k} - \binom{n-1}{k-1} < 0 \), then it holds that

\[
\sum \left( \text{sum for all choices of k minus signs} \right) (\cdots) = (\binom{n-1}{k} - \binom{n-1}{k-1}) \times (\| x_1 \| + \| x_2 \| + \cdots + \| x_n \|).
\]

Therefore it follows that

the right-hand side \( \geq \frac{1}{2^n} \cdot 2^n (1 + (n - 2) + (\binom{n-1}{2} - \binom{n-1}{1}) + (\binom{n-1}{3} - \binom{n-1}{2}) + \cdots + (\binom{n-1}{n-1} - \binom{n-1}{n-2})) \times (\| x_1 \| + \| x_2 \| + \cdots + \| x_n \|)) \)

\[
= \frac{1}{2^{n-1} \cdot \binom{n-1}{k}} \cdot \sum_{n=1}^{n} \| x_1 \|,
\]

where \( k \) is the maximum value that satisfies \( \binom{n-1}{k} - \binom{n-1}{k-1} \geq 0 \), and this is given by \( k = \left\lfloor \frac{n}{2} \right\rfloor \). In \( L^1[0, 1] \), if we set \( x_1 = \cdots = x_n = 1 \), then it holds the equality. Hence this constant is best possible. This completes the proof.
4. Another extension

Adamović (see [3]) has established the following inequality in $\mathbb{R}^n$:

$$\left\| \sum_{i=1}^{n} x_i \right\| + (n-2) \sum_{i=1}^{n} \left\| x_i \right\| \geq \sum_{1 \leq i < j \leq n} \left\| x_i + x_j \right\| \quad \text{for } x_1, \ldots, x_n \in \mathbb{R}^n,$$

Remark that this inequality implies the Hlawka's inequality as a special case of $n = 3$. We shall extend this inequality for a class of Banach spaces. Our proof is based on the simple induction arguments which is essentially due to Vasić [4].

**Theorem 2.** Let $E$ be a Banach space. Suppose that for every $x, y, z \in E$ it holds that

$$\|x + y + z\| + \|x\| + \|y\| + \|z\| \geq \|x + y\| + \|y + z\| + \|z + x\|$$

(the Hlawka's 3-element inequality in $E$).

Then it also holds that

$$\left\| \sum_{i=1}^{n} x_i \right\| + (n-2) \sum_{i=1}^{n} \left\| x_i \right\| \geq \sum_{1 \leq i < j \leq n} \left\| x_i + x_j \right\|$$

for every $n$ and $x_1, \ldots, x_n \in E$.

**Proof.** We prove by induction.

(i) The case $n = 3$ is in our assumption.

(ii) We suppose that the inequality for $n$ holds. Then in the case of $n + 1$, we have

$$\|x_1 + \cdots + x_{n-1} + (x_n + x_{n+1})\| + (n-2)(\|x_1\| + \cdots + \|x_{n-1}\| + \|x_n + x_{n+1}\|)$$

$$\geq \sum_{1 \leq i < j \leq n-1} \left\| x_i + x_j \right\| + \sum_{i=1}^{n-1} \left\| x_i + (x_n + x_{n+1}) \right\|$$

$$\geq \sum_{1 \leq i < j \leq n-1} \left\| x_i + x_j \right\| + \sum_{i=1}^{n-1} (\left\| x_i + x_n \right\| + \left\| x_i + x_{n+1} \right\| + \left\| x_n + x_{n+1} \right\| - \left\| x_i \right\| - \left\| x_n \right\| - \left\| x_{n+1} \right\|)$$

$$= \sum_{1 \leq i < j \leq n-1} \left\| x_i + x_j \right\| + (n-2)\|x_n + x_{n+1}\| - \sum_{i=1}^{n-1} \|x_i\| - (n-1)(\|x_n\| + \|x_{n+1}\|),$$

where we have used the case $n = 3$ for the term $\|x_i + x_n + x_{n+1}\|$. So it follows that

$$\|x_1 + \cdots + x_{n-1} + x_n + x_{n+1}\| + (n-1)(\|x_1\| + \cdots + \|x_{n-1}\| + \|x_n\| + \|x_{n+1}\|)$$

$$\geq \sum_{1 \leq i < j \leq n} \|x_i + x_j\|.$$

This completes the proof.
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