Bull. Kyushu Inst. Tech.
Pure Appl. Math. No. 45, 1998, pp. 9–15

GENERALIZATIONS OF THE HLAWKA'S INEQUALITY

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This paper is dedicated to the late Professor Shigeru Itoh

(Received November 27, 1997)

1. Introduction

We shall extend the Hlawka's 3-element inequality in R^n (or in Hilbertian space) to the n-element inequality in L^1 related to the Hanner's inequality and to the n-element inequality of another type, which is related closely to the Adamović's inequality.

The original Hlawka's inequality is as follows (the 3-element Hlawka's inequality):

$$\|x+y+z\| + \|x\| + \|y\| + \|z\| \geq \|x+y\| + \|y+z\| + \|z+x\| \qquad \text{for } x, y, z \in \mathbf{R}^n.$$

If we set by $x_1 = (x+y)/2$, $x_2 = (y+z)/2$, $x_3 = (z+x)/2$, then we have the following inequality equivalent to the Hlawka's inequality:

$$\begin{aligned} &\|x_1 + x_2 + x_3\| + \|x_1 + x_2 - x_3\| + \|x_1 - x_2 + x_3\| + \|-x_1 + x_2 + x_3\| \\ &\geq 2(\|x_1\| + \|x_2\| + \|x_3\|). \end{aligned}$$

This inequality is rewritten as $\boldsymbol{E}\left\| \sum_{i=1}^{3} \varepsilon_i x_i \right\| \geq \frac{1}{2} \sum_{i=1}^{3} \|x_i\|$, where ε_i is the Rademacher sequence ($\varepsilon_i = \pm 1$ with probability $\frac{1}{2}$) and $\boldsymbol{E}$ means the expectation. We shall give the following extension:

$$\boldsymbol{E}\left\| \sum_{i=1}^{n} \varepsilon_i x_i \right\| \geq \frac{1}{2^{n-1}} \cdot {}_{n-1}C_{[\frac{n}{2}]} \cdot \sum_{i=1}^{n} \|x_i\| \qquad \text{for } x_1, \ldots, x_n \in \mathbf{L}^1.$$

The constant $\frac{1}{2^{n-1}} \cdot {}_{n-1}C_{[\frac{n}{2}]}$ is best possible.

In the Euclidean space $\mathbf{R}^n$, the Hlawka's inequality is generalized by Adamović as follows:

$$\left\| \sum_{i=1}^{n} x_i \right\| + (n-2) \sum_{i=1}^{n} \|x_1\| \geq \sum_{i \leq i<j \leq n} \|x_i + x_j\| \qquad \text{for } x_1, \ldots, x_n \in \mathbf{R}^n.$$

We shall prove that if the 3-element Hlawka's inequality is valid in the Banach space $\boldsymbol{E}$, then it follows also that $\left\| \sum_{i=1}^{n} x_i \right\| + (n-2) \sum_{i=1}^{n} \|x_i\| \geq \sum_{1 \leq i<j \leq n} \|x_i + x_j\|$ for $x_1, \ldots, x_n \in \boldsymbol{E}$. In particular this n-element inequality is valid in L^1.

2. The Hlawka's inequality and the 3-element Hanner's inequality

Hlawka (see [3]) proved the following inequality: For $x, y, z \in \mathbf{R}^n$, it holds that

$$\|x+y+z\| + \|x\| + \|y\| + \|z\| \geq \|x+y\| + \|y+z\| + \|z+x\|.$$

This inequality follows immediately from the triangular inequality and the following identity ([3]);

$$\begin{aligned}
&(\|x+y+z\| + \|x\| + \|y\| + \|z\| - \|x+y\| - \|y+z\| - \|z+x\|) \\
&\quad \times (\|x+y+z\| + \|x\| + \|y\| + \|z\|) \\
&\quad = (\|y\| + \|z\| - \|y+z\|)(\|x\| - \|y+z\| + \|x+y+z\|) \\
&\quad + (\|z\| + \|x\| - \|z+x\|)(\|y\| - \|z+x\| + \|x+y+z\|) \\
&\quad + (\|x\| + \|y\| - \|x+y\|)(\|z\| - \|x+y\| + \|x+y+z\|) \\
&\geq 0.
\end{aligned}$$

Let (S, Σ, μ) be a measure space. The norm of L^1 is given by $\|x\| = \int |x(t)|\, d\mu(t)$. The n-element Hanner's inequality was obtained in [1], [2]. In the case of L^1, the n-element Hanner's inequality is as follows. Let n be a natural number, $\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n$ be the independent Rademacher sequence and $x_1, x_2, \ldots, x_n \in L^1$. Then it holds that

$$\boldsymbol{E}\left\|\sum_{i=1}^{n} \varepsilon_i x_i\right\| \geq \boldsymbol{E}\left|\sum_{i=1}^{n} \varepsilon_i \|x_i\|\right|.$$

In the case where $n = 3$, by the triangular inequality, the Hanner's inequality implies that

$$\begin{aligned}
&\|x_1 + x_2 + x_3\| + \|x_1 + x_2 - x_3\| + \|x_1 - x_2 + x_3\| + \|-x_1 + x_2 + x_3\| \\
&\quad \geq |\,\|x_1\| + \|x_2\| + \|x_3\|\,| + |\,\|x_1\| + \|x_2\| - \|x_3\|\,| + |\,\|x_1\| - \|x_2\| + \|x_3\|\,| \\
&\qquad + |-\|x_1\| + \|x_2\| + \|x_3\|\,| \\
&\quad \geq |\,\|x_1\| + \|x_2\| + \|x_3\| + \|x_1\| + \|x_2\| - \|x_3\| + \|x_1\| - \|x_2\| + \|x_3\| - \|x_1\| + \|x_2\| + \|x_3\|\,| \\
&\quad = 2(\|x_1\| + \|x_2\| + \|x_3\|).
\end{aligned}$$

If we set by $x = x_1 + x_2 - x_3$, $y = x_1 - x_2 + x_3$, $z = -x_1 + x_2 + x_3$, then it follows that

$$\|x+y+z\| + \|x\| + \|y\| + \|z\| \geq \|x+y\| + \|y+z\| + \|z+x\|.$$

Thus the Hlawka's inequality is derived from the 3-element Hanner's inequality. Hence the Hlawka's inequality holds in L^1. Conversely from the Hlawka's inequality, we obtain the 3-element Hanner's inequality in the following way. First, we have

$$\begin{aligned}&\|x+y+z\|+\|-x+y+z\|+\|x-y+z\|+\|x+y-z\|\\&\geq\|(x+y+z)+(x-y-z)\|+\|(x-y+z)+(x+y-z)\|\\&=\|2x\|+\|2x\|=4\|x\|.\end{aligned}$$

Next, let $u=-x+y+z$, $v=x-y+z$, $w=x+y-z$ and we apply the Hlawka's inequality as follows;

$$\begin{aligned}&\|x+y+z\|+\|-x+y+z\|+\|x-y+z\|+\|x+y-z\|\\&\quad=\|u+v+w\|+\|u\|+\|v\|+\|w\|\\&\quad\geq\|u+v\|+\|v+w\|+\|w+u\|\\&\quad=\|2z\|+\|2x\|+\|2y\|=2(\|x\|+\|y\|+\|z\|).\end{aligned}$$

Hence, it holds that

$$\begin{aligned}&\|x+y+z\|+\|-x+y+z\|+\|x-y+z\|+\|x+y-z\|\\&\quad\geq\begin{cases}4\|x\|\\2(\|x\|+\|y\|+\|z\|)\end{cases}.\end{aligned}\tag{*}$$

We show the 3-element Hanner's inequality:

$$\begin{aligned}&\|x_1+x_2+x_3\|+\|x_1+x_2-x_3\|+\|x_1-x_2+x_3\|+\|-x_1+x_2+x_3\|\\&\quad\geq|\,\|x_1\|+\|x_2\|+\|x_3\|\,|+|\,\|x_1\|+\|x_2\|-\|x_3\|\,|+|\,\|x_1\|-\|x_2\|+\|x_3\|\,|\\&\qquad+|-\|x_1\|+\|x_2\|+\|x_3\|\,|.\end{aligned}$$

We can suppose that $\|x_1\|\geq\|x_2\|\geq\|x_3\|$ without loss of generality. Then the last term is:

$$\begin{aligned}&(\|x_1\|+\|x_2\|+\|x_3\|)+(\|x_1\|+\|x_2\|-\|x_3\|)+(\|x_1\|-\|x_2\|+\|x_3\|)\\&\qquad+|-\|x_1\|+\|x_2\|+\|x_3\|\,|\\&\quad=3\|x_1\|+\|x_2\|+\|x_3\|+|-\|x_1\|+\|x_2\|+\|x_3\|\,|.\end{aligned}$$

Consider the two cases:

(i) if $\|x_1\|\geq\|x_2\|+\|x_3\|$, then

$$\text{right-hand side}=3\|x_1\|+\|x_2\|+\|x_3\|+\|x_1\|-\|x_2\|-\|x_3\|=4\|x_1\|$$

(ii) if $\|x_1\| \le \|x_2\| + \|x_3\|$, then

$$\begin{aligned}\text{right-hand side} &= 3\|x_1\| + \|x_2\| + \|x_3\| - \|x_1\| + \|x_2\| + \|x_3\| \\ &= 2(\|x_1\| + \|x_2\| + \|x_3\|).\end{aligned}$$

In these two cases, the Hanner's 3-element inequality is derived by (*).
Accordingly, the Hlawka's inequality and the 3-element Hanner's inequality are equivalent in L^1.

3. Generalization of Hlawka's inequality

The Hlawka's inequality in L^1 is given by

$$\boldsymbol{E}\left\|\sum_{i=1}^{3} \varepsilon_i x_i\right\| \ge \frac{1}{2}\sum_{i=1}^{3} \|x_i\| \qquad \text{for } x_1, x_2, x_3 \in L^1,$$

where E means the expectation with respect to the Rademacher distribution. We shall extend the Hlawka's inequality naturally as follows. Let $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ be the independent Rademacher sequence and $x_1, x_2, \dots, x_n \in L^1$. We can conjecture the following inequality;

$$\boldsymbol{E}\left\|\sum_{i=1}^{n} \varepsilon_i x_i\right\| \ge K(n) \cdot \sum_{i=1}^{n} \|x_i\|,$$

where $K(n)$ is a constant which depends on n.

THEOREM 1. Let n be a natural number, $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n$ be the independent Rademacher sequence and $x_1, x_2, \dots, x_n$ be functions in L^1, then it holds that

$$\boldsymbol{E}\left\|\sum_{i=1}^{n} \varepsilon_i x_i\right\| \ge \frac{1}{2^{n-1}} \cdot {}_{n-1}C_{[\frac{n}{2}]} \cdot \sum_{i=1}^{n} \|x_i\|,$$

where the constant $\frac{1}{2^{n-1}} \cdot {}_{n-1}C_{[\frac{n}{2}]}$ is best possible.

PROOF. We shall start from the n-element Hanner's inequality. We have

$$\begin{aligned}\boldsymbol{E}\left\|\sum_{i=1}^{n} \varepsilon_i x_i\right\| &\ge \boldsymbol{E}\left|\sum_{i=1}^{n} \varepsilon_i \|x_i\|\right| \\ &= \frac{1}{2^n} \cdot \sum_{\left(\substack{\text{sum for all choices} \\ \text{of } \pm \text{ signs}}\right)} |\pm \|x_1\| \pm \|x_2\| \pm \cdots \pm \|x_n\||\end{aligned}$$

$$= \frac{1}{2^n} \left| \|x_1\| + \|x_2\| + \cdots + \|x_n\| \right|$$

$$+ \sum_{\left(\substack{\text{sum for all choices of} \\ \text{one minus signs}}\right)} \left| \|x_1\| + \cdots - \|x_i\| + \cdots + \|x_n\| \right|$$

$$+ \sum_{\left(\substack{\text{sum for all choices of} \\ \text{two minus signs}}\right)} \left| \|x_1\| + \cdots - \|x_j\| + \cdots - \|x_k\| + \cdots + \|x_n\| \right|$$

$$+ \cdots + \left| -\|x_1\| - \|x_2\| - \cdots - \|x_n\| \right|.$$

And we use the triangular inequality $\sum|(\cdots)| \geq |\sum(\cdots)|$. Among the terms

$$\underbrace{\left| \pm \|x_1\| \pm \cdots \pm \|x_i\| \pm \cdots \pm \|x_n\| \right|}_{k \text{ minus signs}}$$

with k minus signs, there are ${}_{n-1}C_k$ terms in which the coefficient of $\|x_i\|$ is $+1$, and there are ${}_{n-1}C_{k-1}$ terms in which the coefficient of $\|x_i\|$ is -1. If ${}_{n-1}C_k - {}_{n-1}C_{k-1} \geq 0$, then it holds that

$$\sum_{\left(\substack{\text{sum for all choices of} \\ k \text{ minus signs}}\right)} (\cdots) = ({}_{n-1}C_k - {}_{n-1}C_{k-1}) \times (\|x_1\| + \|x_2\| + \cdots + \|x_n\|).$$

If ${}_{n-1}C_k - {}_{n-1}C_{k-1} < 0$, then it holds that

$$\sum_{\left(\substack{\text{sum for all choices of} \\ k \text{ minus signs}}\right)} (\cdots) = ({}_{n-1}C_{k-1} - {}_{n-1}C_k) \times (\|x_1\| + \|x_2\| + \cdots + \|x_n\|).$$

Therefore it follows that

$$\text{the right-hand side} \geq \frac{1}{2^n} \cdot 2\{1 + (n-2) + ({}_{n-1}C_2 - {}_{n-1}C_1) + ({}_{n-1}C_3 - {}_{n-1}C_2) + \cdots$$

$$+ ({}_{n-1}C_k - {}_{n-1}C_{k-1})\}(\|x_1\| + \|x_2\| + \cdots + \|x_n\|)$$

$$= \frac{1}{2^{n-1}} \cdot {}_{n-1}C_k \cdot \sum_{n=1}^{n} \|x_1\|,$$

where k is the maximum value that satisfies ${}_{n-1}C_k - {}_{n-1}C_{k-1} \geq 0$, and this is given by $k = \left[\frac{n}{2}\right]$. In $L^1[0,1]$, if we set $x_1 = \cdots = x_n = 1$, then it holds the equality. Hence this constant is best possible. This completes the proof.

4. Another extension

Adamović (see [3]) has established the following inequality in $\mathbf{R}^n$:

$$\left\|\sum_{i=1}^{n} x_i\right\| + (n-2)\sum_{i=1}^{n}\|x_i\| \geq \sum_{1\leq i<j\leq n}\|x_i + x_j\| \qquad \text{for } x_1,\dots,x_n \in \mathbf{R}^n,$$

Remark that this inequality implies the Hlawka's inequality as a special case of $n=3$. We shall extend this inequality for a class of Banach spaces. Our proof is based on the simple induction arguments which is essentially due to Vasić [4].

THEOREM 2. Let E be a Banach space. Suppose that for every $x, y, z \in E$ it holds that

$$\|x+y+z\| + \|x\| + \|y\| + \|z\| \geq \|x+y\| + \|y+z\| + \|z+x\|$$

(the Hlawka's 3-element inequality in E).

Then it also holds that

$$\left\|\sum_{i=1}^{n} x_i\right\| + (n-2)\sum_{i=1}^{n}\|x_i\| \geq \sum_{1\leq i<j\leq n}\|x_i + x_j\|$$

for every n and $x_1,\dots,x_n \in E$.

PROOF. We prove by induction.

(i) The case $n=3$ is in our assumption.

(ii) We suppose that the inequality for n holds. Then in the case of $n+1$, we have

$$\begin{aligned}
&\|x_1 + \cdots + x_{n-1} + (x_n + x_{n+1})\| + (n-2)(\|x_1\| + \cdots + \|x_{n-1}\| + \|x_n + x_{n+1}\|) \\
&\geq \sum_{1\leq i<j\leq n-1}\|x_i + x_j\| + \sum_{i=1}^{n-1}\|x_i + (x_n + x_{n+1})\| \\
&\geq \sum_{1\leq i<j\leq n-1}\|x_i + x_j\| + \sum_{i=1}^{n-1}(\|x_i + x_n\| + \|x_i + x_{n+1}\| + \|x_n + x_{n+1}\| - \|x_i\| - \|x_n\| - \|x_{n+1}\|) \\
&= \sum_{1\leq i<j\leq n+1}\|x_i + x_j\| + (n-2)\|x_n + x_{n+1}\| - \sum_{i=1}^{n-1}\|x_i\| - (n-1)(\|x_n\| + \|x_{n+1}\|),
\end{aligned}$$

where we have used the case $n=3$ for the term $\|x_i + x_n + x_{n+1}\|$. So it follows that

$$\begin{aligned}
&\|x_1 + \cdots + x_{n-1} + x_n + x_{n+1}\| + (n-1)(\|x_1\| + \cdots + \|x_{n-1}\| + \|x_n\| + \|x_{n+1}\|) \\
&\geq \sum_{1\leq i<j\leq n+1}\|x_i + x_j\|.
\end{aligned}$$

This completes the proof.

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