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THE EQUIVALENCE OF THE RADEMACHER TRANSLATIONS OF AN INFINITE PRODUCT MEASURE

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1. Introduction

Let $X = \{X_n\}$ be an i.i.d. random sequence with common density function f and let $\varepsilon = \{\varepsilon_n\}$ be the i.i.d. Rademacher sequence, that is, $\varepsilon_n = \pm 1$ with probability $1/2$, which is independent of X. For a real sequence $a = \{a_n\}$, we set $a\varepsilon = \{a_n \varepsilon_n\}$. The equivalence of the distributions of X and $X + a\varepsilon$ was investigated completely in Sato and Watari [4], Okazaki [2] and Okazaki and Sato [3]. In this paper, we shall consider the equivalence of the distributions of $X + a\varepsilon$ and $X + b\varepsilon$. We say $X + a\varepsilon$ the Rademacher translation of X. By the Kakutani's dichotomy theorem (Kakutani [1]), the distributions of $X + a\varepsilon$ and $X + b\varepsilon$ are either equivalent or singular, and they are equivalent if and only if

$$\sum_{n=1}^{\infty} \int \left| \sqrt{\frac{f(x+a_n)+f(x-a_n)}{2}} - \sqrt{\frac{f(x+b_n)+f(x-b_n)}{2}} \right|^2 dx$$

$$= 2\sum_{n=1}^{\infty} \left(1 - \int \sqrt{\frac{f(x+a_n)+f(x-a_n)}{2}} \sqrt{\frac{f(x+b_n)+f(x-b_n)}{2}}\, dx \right) < \infty.$$

Remark that the density of the distribution of $X_n + a_n\varepsilon_n$ is $(f(x+a_n)+f(x-a_n))/2$.

2. Main results

LEMMA 1. Let $f(x)$ be a density function and $u > 0$, $v > 0$. If it holds that

$$f(x+u) + f(x-u) = f(x+v) + f(x-v), \text{ almost everywhere } (x),$$

then it follows that $u = v$.

PROOF. Firstly we show that

$$f(x+2^n u) + f(x-2^n u) = f(x+2^n v) + f(x-2^n v), \text{ a.e. } (x)$$

for every $n = 1, 2, \cdots$. By the translation of the equality

$$f(x+u) + f(x-u) = f(x+v) + f(x-v), \text{ a.e. } (x)$$

by $\pm u$ and $\pm v$, we have

$$\text{(A)}\quad \begin{cases} f(x+2u)+f(x)=f(x+u+v)+f(x+u-v), \text{ a.e. } (x), \text{ and} \\ f(x)+f(x-2u)=f(x-u+v)+f(x-u-v), \text{ a.e. } (x), \end{cases}$$

$$\text{(B)}\quad \begin{cases} f(x+2v)+f(x)=f(x+v+u)+f(x+v-u), \text{ a.e. } (x), \text{ and} \\ f(x)+f(x-2v)=f(x-v+u)+f(x-v-u), \text{ a.e. } (x). \end{cases}$$

The sum of four functions in the right hand side in (A) coincides with that of in (B), so we obtain $f(x+2u)+2f(x)+f(x-2u)=f(x+2v)+2f(x)+f(x-2v)$, that is, $f(x+2u)+f(x-2u)=f(x+2v)+f(x-2v)$, a.e. (x), which proves the case $n=1$. We set $U=2u$ and $V=2v$. Then similarly we have $f(x+2U)+f(x-2U)=f(x+2V)+f(x-2V)$, a.e. (x), which is the equality desired for $n=2$. By induction, we have the equality for every n.

By the first step, we obtain

$$\begin{aligned} 2 &= \int \sqrt{f(x+2^n u)+f(x-2^n u)}\,\sqrt{f(x+2^n v)+f(x-2^n v)}\,dx \\ &\leqq \int \sqrt{f(x+2^n u)}\,\sqrt{f(x+2^n v)}\,dx + \int \sqrt{f(x+2^n u)}\,\sqrt{f(x-2^n v)}\,dx \\ &\quad + \int \sqrt{f(x-2^n u)}\,\sqrt{f(x+2^n v)}\,dx + \int \sqrt{f(x-2^n u)}\,\sqrt{f(x-2^n v)}\,dx. \end{aligned}$$

Then it holds that

$$\text{the first term} = \int \sqrt{f(x)}\,\sqrt{f(x+2^n(u-v))}\,dx,$$

$$\text{the second term} = \int \sqrt{f(x)}\,\sqrt{f(x-2^n(u+v))}\,dx \to 0 \text{ as } n \to \infty,$$

$$\text{the third term} = \int \sqrt{f(x)}\,\sqrt{f(x+2^n(u+v))}\,dx \to 0 \text{ as } n \to \infty, \text{ and}$$

$$\text{the last term} = \int \sqrt{f(x)}\,\sqrt{f(x+2^n(u-v))}\,dx.$$

If $u \neq v$, then all four terms converges to 0, which is a contradiction. This proves the Lemma 1.

THEOREM 1. If the distributions of $X + a\varepsilon$ and $X + b\varepsilon$ are equivalent, then it follows that $(|a_n| - |b_n|) \in c_0$, that is, $|a_n| - |b_n| \to 0$ as $n \to \infty$.

PROOF. CLAIM 1: $(|a_n| - |b_n|) \in \ell_\infty$.

Without loss of generality, we may suppose that $a_n > 0$ and $b_n > 0$. Taking a subsequence, we suppose in advance that $a_n - b_n \to \infty$. Set $k_n = (a_n - b_n)/2$. Denote by $\lambda = f(x)dx$ the distribution of X_1. Let N_1 be an infinite subset of natural numbers $N = \{1, 2, \cdots\}$ satisfying that

$$\sum_{j \in N_1} (1 - \lambda([-k_j, 2b_j + k_j])) < \infty,$$
$$\sum_{j \in N_1} (1 - \lambda([-2b_j - k_j, k_j])) < \infty,$$
$$\lambda([k_j, \infty)) < 1/2, \text{ and}$$
$$\lambda((-\infty, -k_j)) < 1/2.$$

Consider the subset $\Delta = \prod_{j \in N_1} [-b_j - k_j, b_j + k_j] \times R^{N-N_1}$. Then it follows that

$$P(X + a\varepsilon \in \Delta) = \bigotimes_{n=1}^{\infty} \left[\lambda * \left(\frac{\delta_{a_n} + \delta_{-a_n}}{2} \right) \right] (\Delta)$$
$$= \prod_{j \in N_1} \{\lambda([-b_j - k_j + a_j, b_j + k_j + a_j])/2 + \lambda([-b_j - k_j - a_j, b_j + k_j - a_j])/2\}$$
$$\leqq \prod_{j \in N_1} \{\lambda([k_j, \infty))/2 + \lambda((-\infty, -k_j])/2\} \leqq \prod_{j \in N_1} 1/2 = 0.$$

On the other hand we have

$$P(X + b\varepsilon \in \Delta) = \bigotimes_{n=1}^{\infty} \left[\lambda * \left(\frac{\delta_{b_n} + \delta_{-b_n}}{2} \right) \right] (\Delta)$$
$$= \prod_{j \in N_1} \{\lambda([-k_j, 2b_j + k_j])/2 + \lambda([-2b_j - k_j, k_j])/2\} > 0,$$

which contradicts to the equivalence of the distributions of $X + a\varepsilon$ and $X + b\varepsilon$.

CLAIM 2: $(|a_n| - |b_n|) \in c_0$.

We may assume that $a_n > 0$, $b_n > 0$ since the Rademacher sequence ε is symmetric. By Claim 1, taking a subsequence, we can assume that $a_n - b_n \to h \neq 0$ as $n \to \infty$. Without loss of generality, we suppose that $h > 0$, that is, $a_n > b_n > 0$ and $a_n - b_n \to h > 0$. We devide the proof into two cases as follows.

(1) The case $\{a_n\}$ has a convergent subsequence:

Taking an infinite subset $N_1 \subset N = \{1, 2, \cdots\}$, we suppose that $a_j \to \alpha > 0$ as $j \to \infty$ in $j \in N_1$. Then it follows that $b_j \to \alpha - h \geqq 0$. By the Kakutani's theorem,

$$\int \sqrt{\frac{f(x + a_j) + f(x - a_j)}{2}} \sqrt{\frac{f(x + b_j) + f(x - b_j)}{2}} \, dx \to 1 \ (j \to \infty),$$

which implies that

$$\int \sqrt{\frac{f(x+\alpha)+f(x-\alpha)}{2}} \sqrt{\frac{f(x+\alpha-h)+f(x-\alpha+h)}{2}}\, dx = 1,$$

that is,

$$\int \left| \sqrt{\frac{f(x+\alpha)+f(x-\alpha)}{2}} - \sqrt{\frac{f(x+\alpha-h)+f(x-\alpha+h)}{2}} \right|^2 dx = 0.$$

Thus we obtain that

$$\frac{f(x+\alpha)+f(x-\alpha)}{2} = \frac{f(x+\alpha-h)+f(x-\alpha+h)}{2}, \text{ almost everywhere } (x).$$

By Lemma 1, it follows that $h=0$, which is a contradiction.

(2) The case $\{a_n\}$ does not contain a convergent subsequence:

In this case, there exits an infinite subset N_1 of natural numbers $N = \{1, 2, \cdots\}$ such that $a_j \to \infty$ as $j \to \infty$ $(j \in N_1)$. So it holds that $b_j \to \infty$ and $a_j - b_j \to h > 0$. By the Kakutani's theorem,

$$L_j = \int \sqrt{\frac{f(x+a_j)+f(x-a_j)}{2}} \sqrt{\frac{f(x+b_j)+f(x-b_j)}{2}}\, dx \to 1 \ (j \to \infty).$$

On the other hand, by the manner same to the case (1), we have

$$L_j \leqq \frac{1}{2}\int \sqrt{f(x+a_j)}\sqrt{f(x+b_j)}\, dx + \frac{1}{2}\int \sqrt{f(x+a_j)}\sqrt{f(x-b_j)}\, dx$$
$$+ \frac{1}{2}\int \sqrt{f(x-a_j)}\sqrt{f(x+b_j)}\, dx + \frac{1}{2}\int \sqrt{f(x-a_j)}\sqrt{f(x-b_j)}\, dx.$$

The convergences of these four terms are as follows.

The first term $= \frac{1}{2}\int \sqrt{f(x)}\sqrt{f(x-a_j+b_j)}\, dx \to \frac{1}{2}\int \sqrt{f(x)}\sqrt{f(x-h)}\, dx$,

The second term $= \frac{1}{2}\int \sqrt{f(x)}\sqrt{f(x-a_j-b_j)}\, dx \to 0$,

The third term $= \frac{1}{2}\int \sqrt{f(x)}\sqrt{f(x+a_j+b_j)}\, dx \to 0$, and

The last term $= \frac{1}{2}\int \sqrt{f(x)}\sqrt{f(x+a_j-b_j)}\, dx \to \frac{1}{2}\int \sqrt{f(x)}\sqrt{f(x+h)}\, dx$.

Consequently we have

$$1 \leqq \frac{1}{2}\int\sqrt{f(x)}\sqrt{f(x-h)}\,dx + \frac{1}{2}\int\sqrt{f(x)}\sqrt{f(x+h)}\,dx = \int\sqrt{f(x)}\sqrt{f(x+h)}\,dx$$
$$\leqq \left(\int f(x)\,dx\right)^{1/2}\cdot\left(\int f(x+h)\,dx\right)^{1/2} = 1,$$

which implies that $\int\sqrt{f(x)}\sqrt{f(x+h)}\,dx = 1$ and $\int|\sqrt{f(x)} - \sqrt{f(x+h)}|^2\,dx = 0$. So we have $f(x) = f(x+h)$, and hence $f(x) = f(x \pm nh)$ for every $n = 1, 2, \cdots$. This means that $h = 0$, which is a contradiction. This completes the proof.

THEOREM 2. Suppose that $a \in \ell_\infty$ and that the distributions of $X + a\varepsilon$ and $X + b\varepsilon$ are equivalent. Then it holds that $\sum_{n=1}^{\infty}(a_n^2 - b_n^2)^2 < \infty$.

PROOF. We can assume that $a_n > 0$ and $b_n > 0$ by the symmetricity of ε. By the Kakutani's theorem, it holds that

$$\sum_{n=1}^{\infty}\int\left|\sqrt{\frac{f(x+a_n)+f(x-a_n)}{2}} - \sqrt{\frac{f(x+b_n)+f(x-b_n)}{2}}\right|^2$$
$$= \sum_{n=1}^{\infty}(a_n^2 - b_n^2)^2\int\left|\frac{\sqrt{\frac{f(x+a_n)+f(x-a_n)}{2}} - \sqrt{\frac{f(x+b_n)+f(x-b_n)}{2}}}{a_n^2 - b_n^2}\right|^2 dx < \infty.$$

We set

$$B_n = \int\left|\frac{\sqrt{\frac{f(x+a_n)+f(x-a_n)}{2}} - \sqrt{\frac{f(x+b_n)+f(x-b_n)}{2}}}{a_n^2 - b_n^2}\right|^2 dx.$$

If we show $\liminf_{n\to\infty} B_n > 0$, then it follows that $\sum_{n=1}^{\infty}(a_n^2 - b_n^2)^2 < \infty$. We set

$$F_n(x) = \frac{f(x+a_n)+f(x-a_n)-f(x+b_n)-f(x-b_n)}{a_n^2 - b_n^2},$$
$$G_n(x) = \sqrt{\frac{f(x+a_n)+f(x-a_n)}{2}} + \sqrt{\frac{f(x+b_n)+f(x-b_n)}{2}}, \text{ and}$$
$$D_n(x) = f(x+a_n)+f(x-a_n)-f(x+b_n)-f(x-b_n).$$

Then we have $B_n = (1/4)\int|F_n(x)|^2/G_n(x)^2\,dx$. By the Taylor's expansion, we have

$$D(x) = (f(x+a_n) - f(x+b_n)) - (f(x-b_n) - f(x-a_n))$$
$$= \int_0^1 f'(x+b_n+t(a_n-b_n))\,dt\cdot(a_n-b_n) - \int_0^1 f'(x-a_n+t(a_n-b_n))\,dt\cdot(a_n-b_n)$$

$$= \int_0^1 [f'(x + b_n + t(a_n - b_n)) - f'(x - a_n + t(a_n - b_n))]\, dt \cdot (a_n - b_n)$$
$$= \int_0^1 [\int_0^1 f''(x - a_n + t(a_n - b_n) + s(a_n + b_n))\, ds \cdot (a_n + b_n)]\, dt \cdot (a_n - b_n)$$
$$= \int_0^1 \int_0^1 f''(x - a_n + t(a_n - b_n) + s(a_n + b_n))\, dsdt \cdot (a_n^2 - b_n^2),$$

where f' and f'' are the distributional derivatives. So it follows that

$$F_n(x) = \frac{D(x)}{a_n^2 - b_n^2} = \int_0^1 \int_0^1 f''(x - a_n + t(a_n - b_n) + s(a_n + b_n))\, dsdt, \text{ and}$$
$$F_n(x + a_n) = \int_0^1 \int_0^1 f''(x + t(a_n - b_n) + s(a_n + b_n))\, dsdt.$$

If $\liminf_{n\to\infty} B_n = 0$, then there exists an infinite subset $N_1 \subset N$ such that

$$B_j = (1/4)\int |F_j(x)|^2/G_j(x)^2\, dx = (1/4)\int |F_j(x + a_j)|^2/G_j(x + a_j)^2\, dx \to 0 \text{ as } j \to \infty\ (j \in N_1).$$

This implies that the functions $F_j(x + a_j)/G_j(x + a_j)$ converges to 0 in $L^2(dx)$. Since the functions $G_j(x + a_j)$ are bouded in $L^2(dx)$ with norm $(\int G_j(x + a_j)^2\, dx)^{1/2} \leqq 2$, it follows that $F_j(x + a_j)$ converges to 0 in $L^1(dx)$. Since $(a_j) \in \ell_\infty$, we may assume that $a_j \to \alpha$ as $j \to \infty$ $(j \in N_1)$. By Theorem 1, it holds also that $b_j \to \alpha$. For every infinitely differentiable function g with compact support, we have

$$\begin{aligned}
\int F_j(x + a_j) g(x)\, dx &= \int [\int_0^1 \int_0^1 f''(x + t(a_j - b_j) + s(a_j + b_j))\, dsdt]\, g(x)\, dx \\
&= \int f(x) [\int_0^1 \int_0^1 g''(x - t(a_j - b_j) - s(a_j + b_j))\, dsdt]\, dx \\
&\to \int f(x) [\int_0^1 \int_0^1 g''(x - 2\alpha s)\, dsdt]\, dx \\
&= \int f(x) [\int_0^1 g''(x - 2\alpha s)\, ds]\, dx \\
&= \int [\int_0^1 f(x + 2\alpha s)\, ds]\, g''(x)\, dx = 0.
\end{aligned}$$

Let $h(x) = \int_0^1 f(x + 2\alpha s)\, ds$. Then $h(x)$ is a probability density function and $\int h(x) g''(x)\, dx = 0$ for every infinitely differentiable function g with compact support. This means that the distributional second derivative of $h(x)$ is 0, hence $h(x)$ is a linear function. This is a contradiction.

REMARK 1. In Theorem 2, if $\liminf_{n\to\infty} |a_n| > 0$, then the condition $\Sigma(a_n^2 - b_n^2)^2 < \infty$ is equivalent to $\Sigma(|a_n| - |b_n|)^2 < \infty$ since $\Sigma(a_n^2 - b_n^2)^2 = \Sigma(|a_n| - |b_n|)^2(|a_n| + |b_n|)^2$ and $\liminf_{n\to\infty} |b_n| > 0$ by Theorem 1.

THEOREM 3. Suppose that $a_n \to \infty$ and the distributions of $X + a\varepsilon$ and $X + b\varepsilon$ are equivalent. Then it follows that $\sum_{n=1}^{\infty} (a_n - b_n)^2 < \infty$.

PROOF. Since $a_n \to \infty$, we may assume that $a_n > 0$, $b_n > 0$ and $b_n \to \infty$ by Theorem 1. By the Kakutani's theorem, it holds that

$$\sum_{n=1}^{\infty} (a_n - b_n)^2 \int \left| \frac{\sqrt{\frac{f(x + a_n) + f(x - a_n)}{2}} - \sqrt{\frac{f(x + b_n) + f(x - b_n)}{2}}}{a_n - b_n} \right|^2 dx < \infty.$$

We set

$$C_n = \int \left| \frac{\sqrt{\frac{f(x + a_n) + f(x - a_n)}{2}} - \sqrt{\frac{f(x + b_n) + f(x - b_n)}{2}}}{a_n - b_n} \right|^2 dx$$

and we show that $\liminf_{n\to\infty} C_n > 0$ which implies $\sum_{n=1}^{\infty} (a_n - b_n)^2 < \infty$. Suppose that $\liminf_{n\to\infty} C_n = 0$. Then there exists an infinite subset $N_1 \subset N$ such that the subsequence

$$C_j = \frac{1}{4} \int \left| \frac{f(x + a_j) + f(x - a_j) - f(x + b_j) - f(x - b_j)}{a_j - b_j} \right|^2 \frac{1}{G_j(x)^2} dx \to 0 \text{ as } j \to \infty \ (j \in N_1),$$

where $G_j(x)$ is the same one as in the proof of Theorem 2. By the manner same to Theorem 2, it follows that

$$\int \left| \frac{f(x + a_j) + f(x - a_j) - f(x + b_j) - f(x - b_j)}{a_j - b_j} \right| dx \to 0.$$

By the Taylor's expansion, we have

$$\begin{aligned} &\frac{f(x + a_j) + f(x - a_j) - f(x + b_j) - f(x - b_j)}{a_j - b_j} \\ &= \int_0^1 \{f'(x + b_j + t(a_j - b_j)) - f'(x - a_j + t(a_j - b_j)\} dt, \end{aligned}$$

which implies that

$$\begin{aligned} &\int \left| \int_0^1 \{f'(x + b_j + t(a_j - b_j)) - f'(x - a_j + t(a_j - b_j))\} dt \right| dx \\ &= \int \left| \int_0^1 \{f'(x + t(a_j - b_j)) - f'(x - b_j - a_j + t(a_j - b_j))\} dt \right| dx \to 0. \end{aligned}$$

For every u, it follows that

$$\int_{-\infty}^{u} \int_0^1 \{f'(x + t(a_n - b_n)) - f'(x - b_j - a_j + t(a_j - b_j))\}\, dt\, dx$$
$$= \int_0^1 \{f(u + t(a_j - b_j)) - f(u - b_j - a_j + t(a_j - b_j))\}\, dt$$
$$\to f(u) - f(-\infty) = f(u) = 0,$$

which is a contradiction. This completes the proof.

REMARK 2. In Theorems 1-3, we can suppose that the density $f(x)$ is infinitely differentiable. Let $G = (G_n)$ be the i.i.d. Gaussian sequence with mean 0 and variance 1 independent of X and of ε. Then the distributions of $X + G + a\varepsilon$ and $X + G + b\varepsilon$ are equivalent. The density of $X_1 + G_1$ is infinitely differentiable.

LEMMA 2. $\sum_{n=1}^{\infty} (a_n^2 - b_n^2)^2 < \infty$

$\Leftrightarrow$

$\sum_{n=1}^{\infty} (|a_n| - |b_n|)^4 < \infty$ and $\sum_{n=1}^{\infty} (|a_n| - |b_n|)^2 b_n^2 < \infty$.

PROOF. By $(a_n^2 - b_n^2)^2 = (|a_n| - |b_n|)^2(|a_n| + |b_n|)^2$, it follows that $(|a_n| - |b_n|)^4 \leqq (a_n^2 - b_n^2)^2$ and $(|a_n| - |b_n|)^2 b_n^2 \leqq (a_n^2 - b_n^2)^2$. Conversely, if $((|a_n| - |b_n|)^2) = ((|a_n| - |b_n|)|a_n| - (|a_n| - |b_n|)|b_n|) \in \ell_2$ and $((|a_n| - |b_n|)|b_n|) \in \ell_2$, then we have $((|a_n| - |b_n|)|a_n|) \in \ell_2$, which implies that

$$\sum_{n=1}^{\infty} (a_n^2 - b_n^2)^2 \leqq \sum_{n=1}^{\infty} (|a_n| - |b_n|)^2 (2|a_n|^2 + 2|b_n|^2) < \infty.$$

THEOREM 4. Let $\lambda = f(x)dx$ be the distribution of X_1. Suppose that the density f satisfies the following integrability condition

$$\int_{-\infty}^{\infty} \frac{f''(x)^2}{f(x)}\, dx < \infty.$$

Then for every real sequences $a = (a_n)$, $b = (b_n)$ satisfying $\sum_{n=1}^{\infty} (a_n^2 - b_n^2)^2 < \infty$, the distributions of $X + a\varepsilon$ and $X + b\varepsilon$ are equivalent.

PROOF. We shall assume that $a_n \geqq 0$ and $b_n \geqq 0$ by the symmetricity of ε. We must prove the Kakutani's criterion

$$\sum_{n=1}^{\infty} \int \left| \sqrt{\frac{f(x + a_n) + f(x - a)}{2}} - \sqrt{\frac{f(x + b_n) + f(x - b_n)}{2}} \right|^2 dx < \infty.$$

If we set

$$F(t) = \sqrt{\frac{f(x + b + t(a - b)) + f(x - b - t(a - b))}{2}}$$

then we have

$$\sqrt{\frac{f(x+a)+f(x-a)}{2}}-\sqrt{\frac{f(x+b)+f(x-b)}{2}}=F(1)-F(0)=F'(0)+\textstyle\int_0^1(1-t)F''(t)\,dt.$$

Calculating the derivatives we have

$$F'(0)=\frac{f'(x+b)-f'(x-b)}{4\sqrt{\frac{f(x+b)+f(x-b)}{2}}}\cdot(a-b),\ \text{and}$$

$$F''(t)=\frac{(a-b)^2}{4}\cdot\frac{f''(x+b+t(a-b))+f''(x-b-t(a-b))}{\sqrt{\frac{f(x+b+t(a-b))+f(x-b-t(a-b))}{2}}}$$

$$-\frac{(a-b)^2}{4}\cdot\frac{[f'(x+b+t(a-b))-f'(x-b-t(a-b))]^2}{4\left(\sqrt{\frac{f(x+b+t(a-b))+f(x-b-t(a-b))}{2}}\right)^3}.$$

We set

$$G(t)=\frac{f'(x+tb)}{\sqrt{\frac{f(x+tb)+f(x-tb)}{2}}},\quad \text{and}\quad H(t)=\frac{f'(x-tb)}{\sqrt{\frac{f(x+tb)+f(x-tb)}{2}}}.$$

Then we have

$$G'(t)=\frac{f''(x+tb)b}{\sqrt{\frac{f(x+tb)+f(x-tb)}{2}}}-\frac{f'(x+tb)[f'(x+tb)+f'(x-tb)]b}{4\left(\sqrt{\frac{f(x+tb)+f(x-tb)}{2}}\right)^3},\ \text{and}$$

$$H'(t)=\frac{f''(x-tb)(-b)}{\sqrt{\frac{f(x+tb)+f(x-tb)}{2}}}-\frac{f'(x-tb)[f'(x+tb)+f'(x-tb)]b}{4\left(\sqrt{\frac{f(x+tb)+f(x-tb)}{2}}\right)^3}.$$

We have

$$F'(0)=\frac{(a-b)}{4}[(G(1)-G(0))-(H(1)-H(0))]$$

$$=\frac{(a-b)}{4}[\textstyle\int_0^1(G'(t)-H'(t))\,dt]$$

$$=\frac{(a-b)b}{4}\left[\int_0^1\frac{f''(x+tb)+f''(x-tb)}{\sqrt{\frac{f(x+tb)+f(x-tb)}{2}}}dt-\int_0^1\frac{f'(x+tb)^2-f'(x-tb)^2}{4\left(\sqrt{\frac{f(x+tb)+f(x-tb)}{2}}\right)^3}dt\right].$$

So we obtain that

$$\int\left|\sqrt{\frac{f(x+a)+f(x-a)}{2}}-\sqrt{\frac{f(x+b)+f(x-b)}{2}}\right|^2 dx=\int|F(1)-F(0)|^2\,dx$$

$$=\int|F'(0)+\textstyle\int_0^1(1-t)F''(t)\,dt|^2\,dx$$

$$\leqq 2\int|F'(0)|^2\,dx+2\int|\textstyle\int_0^1(1-t)F''(t)\,dt|^2\,dx \qquad (|a+b|^2\leqq 2(a^2+b^2))$$

$$=2\frac{(a-b)^2b^2}{16}\int_R\left|\int_0^1\frac{f''(x+tb)+f''(x-tb)}{\sqrt{\frac{f(x+tb)+f(x-tb)}{2}}}\,dt\right.$$

$$\left.-\int_0^1\frac{f'(x+tb)^2-f'(x-tb)^2}{4\left(\sqrt{\frac{f(x+tb)+f(x-tb)}{2}}\right)^3}\,dt\right|^2 dx$$

$$+2\frac{(a-b)^4}{16}\int_R\left|\int_0^1(1-t)\frac{f''(x+b+t(a-b))+f''(x-b-t(a-b))}{\sqrt{\frac{f(x+b+t(a-b))+f(x-b-t(a-b))}{2}}}\,dt\right.$$

$$\left.-\int_0^1(1-t)\frac{[f'(x+b+t(a-b))-f'(x-b-t(a-b))]^2}{4\left(\sqrt{\frac{f(x+b+t(a-b))+f(x-b-t(a-b))}{2}}\right)^3}\,dt\right|^2 dx$$

$$\leqq\frac{(a-b)^2b^2}{8}\iint\left[\frac{4[f''(x+tb)^2+f''(x-tb)^2]}{\frac{f(x+tb)+f(x-tb)}{2}}+\frac{4[f'(x+tb)^4+f'(x-tb)^4]}{16\left(\frac{f(x+tb)+f(x-tb)}{2}\right)^3}\right]dtdx$$

$$+\frac{(a-b)^4}{8}\iint\left[\frac{4[f''(x+b+t(a-b))^2+f''(x-b-t(a-b))^2]}{\frac{f(x+b+t(a-b))+f(x-b-t(a-b))}{2}}\right.$$

$$\left.+\frac{4[f'(x+b+t(a-b))^4+f'(x-b-t(a-b))^4]}{16\left(\frac{f(x+b+t(a-b))+f(x-b-t(a-b))}{2}\right)^3}\right]dtdx$$

$$\leqq\frac{(a-b)^2b^2}{8}\iint\left\{8\frac{f''(x+tb)^2}{f(x+tb)}+8\frac{f''(x-tb)^2}{f(x-tb)}+2\frac{f'(x+tb)^4}{f(x+tb)^3}+2\frac{f'(x-tb)^4}{f(x-tb)^3}\right\}dxdt$$

$$+\frac{(a-b)^4}{8}\iint\left\{8\frac{f''(x+b+t(a-b))^2}{f(x+b+t(a-b))}+8\frac{f''(x-b-t(a-b))^2}{f(x-b-t(a-b))}\right.$$

$$\left.+2\frac{f'(x+a+t(a-b))^4}{f(x+b+t(a-b))^3}+2\frac{f'(x-b-t(a-b))^4}{f(x-b-t(a-b))^3}\right\}dxdt$$

$$= \left[\frac{(a-b)^2 b^2}{8} + \frac{(a-b)^4}{8}\right]\left(32\int\frac{(f'')^2}{f}dx + 8\int\frac{(f')^4}{f^3}dx\right).$$

Thus we have

$$\Sigma_{n=1}^{\infty}\int\left|\sqrt{\frac{f(x+a_n)+f(x-a_n)}{2}} - \sqrt{\frac{f(x+b_n)+f(x-b_n)}{2}}\right|^2 dx$$

$$\leqq (\Sigma_n(a_n-b_n)^2 b_n^2 + \Sigma_n(a_n-b_n)^4)\cdot\left(4\int\frac{(f'')^2}{f}dx + \int\frac{(f')^4}{f^3}dx\right)$$

$$< \infty,$$

where the finiteness of $\int (f')^4/f^3\,dx$ is due to Sato and Watari [4].

THEOREM 5. If $\int\frac{(f')^2}{f}dx < \infty$, then for every a and b with $\Sigma_n(a_n-b_n)^2 < \infty$, the distributions of $X + a\varepsilon$ and $X + b\varepsilon$ are equivalent.

PROOF. Let $F(t)$ be

$$F(t) = \sqrt{\frac{f(x+b+t(a-b)) + f(x-b-t(a-b))}{2}}.$$

Then

$$F'(t) = \frac{(a-b)}{4}\,\frac{f'(x+b+t(a-b)) - f'(x-b-t(a-b))}{\sqrt{\frac{f(x+b+t(a-b)) + f(x-b-t(a-b))}{2}}}$$

and by the manner same to Theorem 4, we have

$$\Sigma_{n=1}^{\infty}\int\left|\sqrt{\frac{f(x+a_n)+f(x-a_n)}{2}} - \sqrt{\frac{f(x+b_n)+f(x-b_n)}{2}}\right|^2 dx$$

$$= \Sigma_{n=1}^{\infty}\int\left|\int_0^1 \frac{(a_n-b_n)}{4}\cdot\frac{f'(x+b_n+t(a_n-b_n)) - f'(x-b_n-t(a_n-b_n))}{\sqrt{\frac{f(x+b_n+t(a_n-b_n)) + f(x-b_n-t(a_n-b_n))}{2}}}dt\right|^2 dx$$

$$\leqq \Sigma_{n=1}^{\infty}\frac{(a_n-b_n)^2}{16}\iint\frac{2[f'(x+b_n+t(a_n-b_n))^2 + f'(x-b_n-t(a_n-b_n))^2]}{\frac{f(x+b_n+t(a_n-b_n)) + f(x-b_n-t(a_n-b_n))}{2}}dtdx$$

$$\leqq \Sigma_{n=1}^{\infty}\frac{(a_n-b_n)^2}{16}\cdot 4\cdot\int\frac{(f')^2}{f}dx < \infty.$$

REMARK 3. In Theorem 3, we can not conclude that $\Sigma(a_n^2 - b_n^2)^2 < \infty$. In fact, if $\Sigma(a_n^2 - b_n^2)^2 = \infty$, $\Sigma(a_n - b_n)^2 < \infty$ and $\int (f')^2/f\,dx < \infty$, the distributions of $X + a\varepsilon$ and $X + b\varepsilon$ are equivalent.

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