A GENERALIZATION OF HANNER'S INEQUALITY

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1. Introduction

In the preceding paper [2], we extended the Hanner's 2-element inequality in $L^p$ to the $n$-element inequality and determined the type 2 (cotype 2) constant of $L^p$. However the main result in [2] was restricted to the real valued functions in $L^p$ and the general complex case was left open. In this paper, we prove that the $n$-element version of the Hanner's inequality is also valid for the complex valued $L^p$-functions. Let $\epsilon_1, \epsilon_2, \cdots, \epsilon_n$ be the independent Rademacher sequence and $x_1, x_2, \cdots, x_n \in L^p$. We prove that

\[ E \left| \sum_{i=1}^n \epsilon_i x_i \right|^p \geq E \left| \sum_{i=1}^n \epsilon_i \right|^{\frac{p}{2}} \left| x_i \right|^p \quad \text{for } 1 \leq p \leq 2, \text{ and} \]
\[ E \left| \sum_{i=1}^n \epsilon_i x_i \right|^p \leq E \left| \sum_{i=1}^n \epsilon_i \right|^{\frac{p}{2}} \left| x_i \right|^p \quad \text{for } 2 \leq p < \infty. \]

We prove a heredity property of Hanner cotype $p (1 \leq p \leq 2)$. If $X$ is a Banach space of Hanner cotype $p$, then $L^p(X)$ is of Hanner cotype $p$.

2. Hanner's inequality

Let $1 \leq p < \infty$, $(S, \Sigma, \mu)$ be a probability space and $L^p = L^p(S, \Sigma, \mu)$. The norm of $L^p$ is given by $\|x\| = (\int |x(t)|^p d\mu(t))^{1/p}$. Hanner [1] proved the following inequalities. For $x_1, x_2 \in L^p$, it holds that for $1 < p \leq 2$

\[ \|x_1 + x_2\|^p + \|x_1 - x_2\|^p \geq \|x_1\|^p + \|x_2\|^p + \|x_1 - x_2\|^p \]

and for $2 \leq p < \infty$

\[ \|x_1 + x_2\|^p + \|x_1 - x_2\|^p \leq \|x_1\|^p + \|x_2\|^p + \|x_1 - x_2\|^p. \]

In the case where $p = 1$, the Hanner's inequality is just the triangular inequality. The case $p = 2$ is

\[ \|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 \geq \|x_1\|^2 + \|x_2\|^2 + \|x_1 - x_2\|^2 \]

the parallelogram law. The Hanner's inequality is rewritten as follows. Let $\epsilon_1, \epsilon_2$ be the independent Rademacher random variables with the distribution $\epsilon_i = \pm 1$ with probability $1/2$. Then the Hanner's inequality is given by

\[ E \left| \sum_{i=1}^2 \epsilon_i x_i \right|^p \geq E \left| \sum_{i=1}^2 \epsilon_i \right|^{\frac{p}{2}} \left| x_i \right|^p \quad \text{for } 1 < p \leq 2, \text{ and} \]
\[ E \left| \sum_{i=1}^2 \epsilon_i x_i \right|^p \leq E \left| \sum_{i=1}^2 \epsilon_i \right|^{\frac{p}{2}} \left| x_i \right|^p \quad \text{for } 2 \leq p < \infty. \]
where $E$ means the expectation with respect to the Rademacher distribution.

In the preceding paper [2], we extended the Hanner's 2-element inequality to the $n$-element inequality as follows. Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ be the independent Rademacher sequence and $x_1, x_2, \cdots, x_n \in L^p$. Then if each $x_i$ is real valued function, then it holds that

$$E \| \sum_{i=1}^n \varepsilon_i x_i \|^p \geq E \| \sum_{i=1}^n \varepsilon_i x_i \|^p \quad \text{for } 1 \leq p \leq 2, \text{ and}$$

$$E \| \sum_{i=1}^n \varepsilon_i x_i \|^p \leq E \| \sum_{i=1}^n \varepsilon_i x_i \|^p \quad \text{for } 2 \leq p < \infty.$$ 

The general complex valued cases were left open in [2]. In this paper, we show that the Hanner's $n$-element inequality is valid also for complex valued functions $x_1, x_2, \cdots, x_n \in L^p$. To show the general complex case, we use the full real version of the above Hanner's $n$-element inequality.

**Lemma 1.** Let $g_1$ and $g_2$ be the independent Gaussian random variables with mean 0 and variance 1 on a probability space $(\Omega, P)$. Let $\phi : \mathbb{C} \to L^p(\Omega, P; \mathbb{R})$ be, for $z = u + iv \in \mathbb{C}$,

$$\phi(z)(\omega) = c_p(\varepsilon g_1(\omega) + v g_2(\omega)),$$

where $L^p(\Omega, P; \mathbb{R})$ is the real valued $L^p$ space and $c_p$ be the constant $c_p = (\int |g_1(\omega)|^p dP(\omega))^{-1/p}$. Then it hold that

1. $\phi$ is real linear, that is, $\phi(sz_1 + tz_2) = s\phi(z_1) + t\phi(z_2)$ for $z_1, z_2 \in \mathbb{C}$ and $s, t \in \mathbb{R}$, and
2. $\phi$ is isometry, that is,

$$\| \phi(z)(\omega) \|_{L^p(\Omega)} = (\int |\phi(z)(\omega)|^p dP(\omega))^{1/p} = |z| = \sqrt{u^2 + v^2}.$$

**Proof.** 1. is clear. To show 2, we calculate the $L^p$-norm of $\phi(z)$.

$$\| \phi(z) \|^p = c_p^p \left( \int |\phi(\varepsilon g_1(\omega) + v g_2(\omega))|^p dP(\omega) \right)$$

$$= c_p^p \left( \int \frac{u}{\sqrt{u^2 + v^2}} g_1(\omega) + \frac{v}{\sqrt{u^2 + v^2}} g_2(\omega) \right)^p dP(\omega)$$

$$= (\sqrt{u^2 + v^2})^p,$$

where we have used the fact that the distributions of $sg_1 + tg_2$ ($s^2 + t^2 = 1, s, t \in \mathbb{R}$) and $g_1$ are identical, hence the last integral is $c_p^p \cdot P$. This proves the Lemma.

**Lemma 2.** Let $p$ be $1 \leq p < \infty$, $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ be independent Rademacher random variables and $z_1, z_2, \cdots, z_n$ be complex numbers. Then it holds that for $1 \leq p \leq 2$

$$E \left| \sum_{i=1}^n \varepsilon_i z_i \right|^p \geq E \left| \sum_{i=1}^n \varepsilon_i |z_i| \right|^p,$$

and for $2 \leq p < \infty$
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\[ \mathbb{E} \left| \sum_{i=1}^{n} \epsilon_i z_i \right|^p \leq \mathbb{E} \left| \sum_{i=1}^{n} \epsilon_i |z_i|^p \right. \]

**Proof.** Let \( \varphi \) be the mapping given in Lemma 1. We prove only the case \( 1 \leq p \leq 2 \). The case \( 2 \leq p < \infty \) is analogous. We have

\[
\begin{align*}
\mathbb{E} \left| \sum_{i=1}^{n} \epsilon_i z_i \right|^p &= \mathbb{E} \left\| \varphi \left( \sum_{i=1}^{n} \epsilon_i z_i \right) \right\|^p \\
&= \mathbb{E} \left\| \sum_{i=1}^{n} \epsilon_i \varphi(z_i) \right\|^p \\
&\geq \mathbb{E} \left( \sum_{i=1}^{n} \epsilon_i \| \varphi(z_i) \| \right)^p \\
&= \mathbb{E} \left( \sum_{i=1}^{n} \epsilon_i |z_i|^p \right)^p,
\end{align*}
\]

where the above inequality is the Hanner’s inequality for the real \( L^p \)-functions \( \{ \varphi(z_i) \} \) (see [2]) and the last equality follows from Lemma 1.

**Lemma 3 (Hanner [1]).** Let \( x \geq 0 \) and \( u \geq 0 \). Let \( f(u) \) be

\[ f(u) = |u^{1/p} + x|^p + |u^{1/p} - x|^p. \]

If \( 1 \leq p \leq 2 \), then \( f(u) \) is a convex function, and if \( 2 \leq p < \infty \), then \( f(u) \) is a concave function.

**Lemma 4.** Let \( u_1, u_2, \ldots, u_n \geq 0 \) and let \( F(u_1, u_2, \ldots, u_n) \) be

\[ F(u_1, u_2, \ldots, u_n) = \mathbb{E} \left( \sum_{i=1}^{n} \epsilon_i u_i \right)^p. \]

Then regarding \( F \) as a function of each \( u_i \), \( F \) is convex for \( 1 \leq p \leq 2 \) and \( F \) is concave for \( 2 \leq p < \infty \).

**Proof.** The Lemma follows from Lemma 3. See also Kigami, Okazaki and Takahashi [2].

**Theorem 1.** Let \( n \) be a natural number, \( \epsilon_1, \epsilon_2, \ldots, \epsilon_n \) be independent Rademacher random variables and \( x_1, x_2, \ldots, x_n \) be functions in \( L^p \).

1. If \( 1 \leq p \leq 2 \), then it holds that

\[ \mathbb{E} \left( \sum_{i=1}^{n} \epsilon_i x_i \right)^p \geq \mathbb{E} \left( \sum_{i=1}^{n} \epsilon_i \| x_i \|^p \right). \]

2. If \( 2 \leq p < \infty \), then it holds that

\[ \mathbb{E} \left( \sum_{i=1}^{n} \epsilon_i x_i \right)^p \leq \mathbb{E} \left( \sum_{i=1}^{n} \epsilon_i \| x_i \|^p \right). \]

**Proof.** (1) Suppose that \( 1 \leq p \leq 2 \). By Lemma 2, we have

\[
\begin{align*}
\mathbb{E} \left( \sum_{i=1}^{n} \epsilon_i x_i \right)^p &= \mathbb{E} \left( \sum_{i=1}^{n} \epsilon_i (\omega) x_i(t) \right)^p d\mu(t) \\
&= \int \mathbb{E} \left( \sum_{i=1}^{n} \epsilon_i (\omega) x_i(t) \right)^p d\mu(t) \\
&\geq \int \mathbb{E} \left( \sum_{i=1}^{n} \epsilon_i (\omega) |x_i(t)| \right)^p d\mu(t) \\
&= \mathbb{E} \left( \sum_{i=1}^{n} \epsilon_i |x_i|^p \right),
\end{align*}
\]

(2) If \( 2 \leq p < \infty \), then it holds that

\[ \mathbb{E} \left( \sum_{i=1}^{n} \epsilon_i x_i \right)^p \leq \mathbb{E} \left( \sum_{i=1}^{n} \epsilon_i \| x_i \|^p \right). \]
where $|x_i(t)| = |x_i(t)|$. So we can suppose that each $x_i$ is a non-negative function, $x_i(t) \geq 0$. By Lemma 3 and by the Jensen’s inequality, we obtain that

$$F(x_1(t)^p, x_2(t)^p, \ldots, x_n(t)^p) d\mu(t) \geq F(\int \frac{x_1(t)^p d\mu(t)}{\int \frac{x_i(t)^p d\mu(t)}{\int x_i(t)^p d\mu(t)}}),$$

where $F$ is the function given in Lemma 4. This proves (1).

(2) The case where $2 \leq p < \infty$ is obtained by the manner same to the case (1). In this case, $F$ is concave and we obtain the converse inequality

$$F(x_1(t)^p, x_2(t)^p, \ldots, x_n(t)^p) d\mu(t) \leq F(\int x_1(t)^p d\mu(t), \int x_2(t)^p d\mu(t), \ldots, \int x_n(t)^p d\mu(t)).$$

by the Jensen’s inequality. This completes the proof.

**Remark.** In the case where $p = 1$, Hanner’s 2-element inequality

$$\|x_1 + x_2\| + \|x_1 - x_2\| \geq \|x_1\| + \|x_2\| + \|x_1\| - \|x_2\|$$

is nothing but the triangular inequality. So this 2-element inequality is valid in all Banach spaces. But the $n$-element inequality

$$E \|\sum_{i=1}^n \epsilon_i x_i\|^p \geq E \|\sum_{i=1}^n \epsilon_i \|x_i\||^p$$

is not necessarily valid in all Banach spaces. If this $n$-element inequality is valid for every $n$, then the Banach space is of cotype 2, see [2].

3. Hanner type and Hanner cotype

Let $X$ be a Banach space. Denote by $L^p(X) = L^p(S, \Sigma; \mu; X)$ the Banach space of $X$-valued $L^p$-functions $f(t) : S \to X$ with norm

$$\|f\|_{L^p(X)} = \left(\int_S \|f(t)\|^p d\mu(t)\right)^{1/p}.$$

Let $X$ be a Banach space with norm $\|\|$ . We say that $X$ is of Hanner cotype $p$ (1 $\leq p \leq 2$) if it holds that

$$E \|\sum_{i=1}^n \epsilon_i x_i\|^p \geq E \|\sum_{i=1}^n \epsilon_i \|x_i\||^p$$

for every $n$ and every $x_1, x_2, \ldots, x_n \in X$, where $\{\epsilon_i\}$ are independent Rademacher random variables. We say that $X$ is of Hanner type $p$ (2 $\leq p < \infty$) if it holds that

$$E \|\sum_{i=1}^n \epsilon_i x_i\|^p \leq E \|\sum_{i=1}^n \epsilon_i \|x_i\||^p$$

for every $n$ and every $x_1, x_2, \ldots, x_n \in X$. By Theorem 1, $L^p$ is of Hanner cotype $p$ for $1 \leq p \leq 2$ and of Hanner type $p$ for $2 \leq p < \infty$. 
THEOREM 2. If $X$ is a Banach space of Hanner cotype $p$ (resp., Hanner type $p$), then $L^p(X)$ is of Hanner cotype $p$ (resp., Hanner type $p$).

PROOF. For $f_1, f_2, \ldots, f_n \in L^p(X)$, we have

$$E \left\| \sum_{i=1}^n \varepsilon_i f_i \right\|_{L^p(X)} = E \left( \int \left\| \sum_{i=1}^n \varepsilon_i f_i(t) \right\|^p d\mu(t) \right)$$

$$= \int \left( E \left\| \sum_{i=1}^n \varepsilon_i f_i(t) \right\|^p \right) d\mu(t)$$

$$\geq \int \left( E \left\| \sum_{i=1}^n \varepsilon_i f_i(t) \right\| \right) d\mu(t)$$

$$= E \left( \int \left\| \sum_{i=1}^n \varepsilon_i f_i(t) \right\| d\mu(t) \right)$$

$$= E \left\| \sum_{i=1}^n \varepsilon_i f_i \right\|_{L^p([0,1])}$$

$$\geq E \left\| \sum_{i=1}^n \varepsilon_i \left\| f_i \right\|_{L^p} \right\|_{L^p([0,1])}$$

where the two inequalities above follow from the fact that $X$ and $L^p(\mathbb{R})$ are of Hanner cotype $p$ (the assumption on $X$ and Theorem 1) and $F_i$ is the real function $F_i(t) = \| f_i(t) \|$. This completes the proof.

PROPOSITION 1. Let $1 \leq p \leq r \leq 2$. Then $L^r$ is of Hanner cotype $p$ and $L^p(L^r)$ is of Hanner cotype $p$.

PROOF. $L^r$ is isometrically imbeddable into $L^p$ since $1 \leq p \leq r \leq 2$, so the assertion follows.

References

[1] O. Hanner, On the uniform convexity of $L^p$ and $\ell^n$, Arkiv för Mat. 3 (1956), 239–244.

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