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# A GENERALIZATION OF HANNER'S INEQUALITY

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## 1. Introduction

In the preceding paper [2], we extended the Hanner's 2-element inequality in $L^p$ to the $n$-element inequality and determined the type 2 (cotype 2) constant of $L^p$. However the main result in [2] was restricted to the real valued functions in $L^p$ and the general complex case was left open. In this paper, we prove that the $n$-element version of the Hanner's inequality is also valid for the complex valued $L^p$-functions. Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ be the independent Rademacher sequence and $x_1, x_2, \cdots x_n \in L^p$. We prove that

$$\mathrm{E}\,\|\textstyle\sum_{i=1}^n \varepsilon_i x_i\|^p \geqq \mathrm{E}\,|\sum_{i=1}^n \varepsilon_i \|x_i\||^p \qquad \text{for } 1 \leqq p \leqq 2, \text{ and}$$
$$\mathrm{E}\,\|\textstyle\sum_{i=1}^n \varepsilon_i x_i\|^p \leqq \mathrm{E}\,|\sum_{i=1}^n \varepsilon_i \|x_i\||^p \qquad \text{for } 2 \leqq p < \infty.$$

We prove a heredity property of Hanner cotype $p(1 \leqq p \leqq 2)$. If $X$ is a Banach space of Hanner cotype $p$, then $L^p(X)$ is of Hanner cotype $p$.

## 2. Hanner's inequality

Let $1 \leqq p < \infty$, $(S, \Sigma, \mu)$ be a probability space and $L^p = L^p(S, \Sigma, \mu)$. The norm of $L^p$ is given by $\|x\| = (\int |x(t)|^p d\mu(t))^{1/p}$. Hanner [1] proved the following inequalities. For $x_1, x_2 \in L^p$, it holds that for $1 < p \leqq 2$

$$\|x_1 + x_2\|^p + \|x_1 - x_2\|^p \geqq |\,\|x_1\| + \|x_2\|\,|^p + |\,\|x_1\| - \|x_2\|\,|^p$$

and for $2 \leqq p < \infty$

$$\|x_1 + x_2\|^p + \|x_1 - x_2\|^p \leqq |\,\|x_1\| + \|x_2\|\,|^p + |\,\|x_1\| - \|x_2\|\,|^p.$$

In the case where $p = 1$, the Hanner's inequality is just the triangular inequality. The case $p = 2$ is

$$\|x_1 + x_2\|^2 + \|x_1 - x_2\|^2 \geqq |\,\|x_1\| + \|x_2\|\,|^2 + |\,\|x_1\| - \|x_2\|\,|^2$$

the parallelogram law. The Hanner's inequality is rewritten as follows. Let $\varepsilon_1, \varepsilon_2$ be the independent Rademacher random variables with the distribution $\varepsilon_i = \pm 1$ with probability $1/2$. Then the Hanner's inequality is given by

$$\mathrm{E}\,\|\textstyle\sum_{i=1}^2 \varepsilon_i x_i\|^p \geqq \mathrm{E}\,|\sum_{i=1}^2 \varepsilon_i \|x_i\||^p \qquad \text{for } 1 < p \leqq 2, \text{ and}$$
$$\mathrm{E}\,\|\textstyle\sum_{i=1}^2 \varepsilon_i x_i\|^p \leqq \mathrm{E}\,|\sum_{i=1}^2 \varepsilon_i \|x_i\||^p \qquad \text{for } 2 \leqq p < \infty,$$

where E means the expectation with respect to the Rademacher distribution.

In the preceding paper [2], we extended the Hanner's 2-element inequality to the $n$-element inequality as follows. Let $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ be the independent Rademacher sequence and $x_1, x_2, \cdots x_n \in L^p$. Then if each $x_i$ is real valued function, then it holds that

$$\mathrm{E}\,\|\textstyle\sum_{i=1}^n \varepsilon_i x_i\|^p \geqq \mathrm{E}\,|\textstyle\sum_{i=1}^n \varepsilon_i \|x_i\||^p \quad \text{for } 1 \leqq p \leqq 2, \text{ and}$$
$$\mathrm{E}\,\|\textstyle\sum_{i=1}^n \varepsilon_i x_i\|^p \leqq \mathrm{E}\,|\textstyle\sum_{i=1}^n \varepsilon_i \|x_i\||^p \quad \text{for } 2 \leqq p < \infty.$$

The general complex valued cases were left open in [2]. In this paper, we show that the Hanner's $n$-element inequality is valid also for complex valued functions $x_1, x_2 \cdots x_n \in L^p$. To show the general complex case, we use the full real version of the above Hanner's $n$-element inequality.

LEMMA 1. Let $g_1$ and $g_2$ be the independent Gaussian random variables with mean 0 and variance 1 on a probability space $(\Omega, P)$. Let $\varphi: \mathbb{C} \to L^p(\Omega, P; \mathbb{R})$ be, for $z = u + iv \in \mathbb{C}$,

$$\varphi(z)(\omega) = c_p(ug_1(\omega) + vg_2(\omega)),$$

where $L^p(\Omega, P; \mathbb{R})$ is the real valued $L^p$ space and $c_p$ be the constant $c_p = (\int |g_1(\omega)|^p dP(\omega))^{-1/p}$. Then it hold that

1. $\varphi$ is real linear, that is, $\varphi(sz_1 + tz_2) = s\varphi(z_1) + t\varphi(z_2)$ for $z_1, z_2 \in \mathbb{C}$ and $s, t \in \mathbb{R}$, and

2. $\varphi$ is isometry, that is,

$$\|\varphi(z)\|_{L^p(\Omega)} = (\int |\varphi(z)(\omega)|^p dP(\omega))^{1/p} = |z| = \sqrt{u^2 + v^2}.$$

PROOF. 1. is clear. To show 2, we calculate the $L^p$-norm of $\varphi(z)$.

$$\begin{aligned} \|\varphi(z)\|^p &= c_p^p \int |\varphi(ug_1(\omega) + vg_2(\omega))|^p dP(\omega) \\ &= c_p^p(\sqrt{u^2 + v^2})^p \int \left| \frac{u}{\sqrt{u^2 + v^2}} g_1(\omega) + \frac{v}{\sqrt{u^2 + v^2}} g_2(\omega) \right|^p dP(\omega) \\ &= (\sqrt{u^2 + v^2})^p, \end{aligned}$$

where we have used the fact that the distributions of $sg_1 + tg_2$ $(s^2 + t^2 = 1,\ s, t \in \mathbb{R})$ and $g_1$ are identical, hence the last integral is $c_p^{-p}$. This proves the Lemma.

LEMMA 2. Let $p$ be $1 \leqq p < \infty$, $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ be independent Rademacher random variables and $z_1, z_2, \cdots, z_n$ be complex numbers. then it holds that for $1 \leqq p \leqq 2$

$$\mathrm{E}\,|\textstyle\sum_{i=1}^n \varepsilon_i z_i|^p \geqq \mathrm{E}\,|\textstyle\sum_{i=1}^n \varepsilon_i |z_i||^p,$$

and for $2 \leqq p < \infty$

$$\mathrm{E}\,|\sum_{i=1}^{n}\varepsilon_i z_i|^p \leqq \mathrm{E}\,|\sum_{i=1}^{n}\varepsilon_i |z_i||^p.$$

PROOF. Let $\varphi$ be the mapping given in Lemma 1. We prove only the case $1 \leqq p \leqq 2$. The case $2 \leqq p < \infty$ is analogous. We have

$$\begin{aligned}\mathrm{E}\,|\sum_{i=1}^{n}\varepsilon_i z_i|^p &= \mathrm{E}\,\|\varphi(\sum_{i=1}^{n}\varepsilon_i z_i)\|^p \\ &= \mathrm{E}\,\|\sum_{i=1}^{n}\varepsilon_i \varphi(z_i)\|^p \\ &\geqq \mathrm{E}\,|\sum_{i=1}^{n}\varepsilon_i \|\varphi(z_i)\||^p \\ &= \mathrm{E}\,|\sum_{i=1}^{n}\varepsilon_i |z_i||^p,\end{aligned}$$

where the above inequality is the Hanner's inequality for the real $L^p$-functions $\{\varphi(z_i)\}$ (see [2]) and the last equality follows from Lemma 1.

LEMMA 3 (Hanner [1]). Let $\alpha \geqq 0$ and $u \geqq 0$. Let $f(u)$ be

$$f(u) = |u^{1/p} + \alpha|^p + |u^{1/p} - \alpha|^p.$$

If $1 \leqq p \leqq 2$, then $f(u)$ is a convex function, and if $2 \leqq p < \infty$, then $f(u)$ is a concave function.

LEMMA 4. Let $u_1, u_2, \cdots, u_n \geqq 0$ and let $F(u_1, u_2, \cdots, u_n)$ be

$$F(u_1, u_2, \cdots, u_n) = \mathrm{E}\,|\sum_{i=1}^{n}\varepsilon_i u_i^{1/p}|^p.$$

Then regarding $F$ as a function of each $u_i$, $F$ is convex for $1 \leqq p \leqq 2$ and $F$ is concave for $2 \leqq p < \infty$.

PROOF. The Lemma follows from Lemma 3. See also Kigami, Okazaki and Takahashi [2].

THEOREM 1. Let $n$ be a natural number, $\varepsilon_1, \varepsilon_2, \cdots, \varepsilon_n$ be independent Rademacher random variables and $x_1, x_2, \cdots, x_n$ be functions in $L^p$.

(1) If $1 \leqq p \leqq 2$, then it holds that

$$\mathrm{E}\,\|\sum_{i=1}^{n}\varepsilon_i x_i\|^p \geqq \mathrm{E}\,|\sum_{i=1}^{n}\varepsilon_i \|x_i\||^p.$$

(2) If $2 \leqq p < \infty$, then it holds that

$$\mathrm{E}\,\|\sum_{i=1}^{n}\varepsilon_i x_i\|^p \leqq \mathrm{E}\,|\sum_{i=1}^{n}\varepsilon_i \|x_i\||^p.$$

PROOF. (1) Suppose that $1 \leqq p \leqq 2$. By Lemma 2, we have

$$\begin{aligned}\mathrm{E}\,\|\sum_{i=1}^{n}\varepsilon_i x_i\|^p &= \mathrm{E}\,(\textstyle\int_s |\sum_{i=1}^{n}\varepsilon_i(\omega) x_i(t)|^p\, d\mu(t)) \\ &= \textstyle\int_s \mathrm{E}\,|\sum_{i=1}^{n}\varepsilon_i(\omega)\, x_i(t)|^p\, d\mu(t) \\ &\geqq \textstyle\int_s \mathrm{E}\,|\sum_{i=1}^{n}\varepsilon_i(\omega)\, |x_i(t)||^p\, d\mu(t) \\ &= \mathrm{E}\,\|\sum_{i=1}^{n}\varepsilon_i |x_i|\|^p,\end{aligned}$$

where $|x_i|(t)=|x_i(t)|$. So we can suppose that each $x_i$ is a non-negative function, $x_i(t)\geqq 0$. By Lemma 3 and by the Jensen's inequality, we obtain that

$$\int_s F(x_1(t)^p, x_2(t)^p, \cdots, x_n(t)^p)\,d\mu(t)$$
$$\geqq F(\textstyle\int_s x_1(t)^p d\mu(t), \int_s x_2(t)^p d\mu(t), \cdots, \int_s x_n(t)^p d\mu(t)),$$

where $F$ is the function given in Lemma 4. This proves (1).

(2) The case where $2\leqq p<\infty$ is obtained by the manner same to the case (1). In this case, $F$ is concave and we obtain the converse inequality

$$\int_s F(x_1(t)^p, x_2(t)^p, \cdots, x_n(t)^p)\,d\mu(t)$$
$$\leqq F(\textstyle\int_s x_1(t)^p\, d\mu(t), \int_s x_2(t)^p\, d\mu(t), \cdots, \int_s x_n(t)^p\, d\mu(t)),$$

by the Jensen's inequality. This completes the proof.

REMARK. In the case where $p=1$, Hanner's 2-element inequality

$$\|x_1+x_2\|+\|x_1-x_2\|\geqq\|x_1\|+\|x_2\|+|\,\|x_1\|-\|x_2\|\,|$$

is nothing but the triangular inequality. So this 2-element inequality is valid in all Banach spaces. But the $n$-element inequality

$$\mathrm{E}\,\|\textstyle\sum_{i=1}^n \varepsilon_i x_i\|^p \geqq \mathrm{E}\,|\sum_{i=1}^n \varepsilon_i\,\|x_i\|\,|^p$$

is not necessarily valid in all Banach spaces. If this $n$-element inequality is valid for every $n$, then the Banach space is of cotype 2, see [2].

## 3. Hanner type and Hanner cotype

Let $X$ be a Banach space. Denote by $L^p(X)=L^p(S, \Sigma, \mu; X)$ the Banach space of $X$-valued $L^p$-functions $f(t)\colon S\to X$ with norm

$$\|f\|_{L^p(X)}=\left(\int_S \|f(t)\|_X^p\, d\mu(t)\right)^{1/p}.$$

Let $X$ be a Banach space with norm $\|\ \|$. We say that $X$ is of Hanner cotype $p$ $(1\leqq p\leqq 2)$ if it holds that

$$\mathrm{E}\,\|\textstyle\sum_{i=1}^n \varepsilon_i x_i\|^p \geqq \mathrm{E}\,|\sum_{i=1}^n \varepsilon_i\,\|x_i\|\,|^p$$

for every $n$ and every $x_1, x_2, \cdots, x_n\in X$, where $\{\varepsilon_i\}$ are independent Rademacher random variables. We say that $X$ is of Hanner type $p$ $(2\leqq p<\infty)$ if it holds that

$$\mathrm{E}\,\|\textstyle\sum_{i=1}^n \varepsilon_i x_i\|^p \leqq \mathrm{E}\,|\sum_{i=1}^n \varepsilon_i\,\|x_i\|\,|^p$$

for every $n$ and every $x_1, x_2, \cdots, x_n\in X$. By Theorem 1, $L^p$ is of Hanner cotype $p$ for $1\leqq p\leqq 2$ and of Hanner type $p$ for $2\leqq p<\infty$.

THEOREM 2. If $X$ is a Banach space of Hanner cotype $p$ (resp., Hanner type $p$), then $L^p(X)$ is of Hanner cotype $p$ (resp., Hanner type $p$).

PROOF. For $f_1, f_2, \cdots, f_n \in L^p(X)$, we have

$$\begin{aligned}
\mathrm{E}\left\|\sum_{i=1}^n \varepsilon_i f_i\right\|_{L^p(X)}^p &= \mathrm{E}\left(\int \left\|\sum_{i=1}^n \varepsilon_i f_i(t)\right\|^p d\mu(t)\right)\\
&= \int \left(\mathrm{E}\left\|\sum_{i=1}^n \varepsilon_i f_i(t)\right\|^p\right) d\mu(t)\\
&\geqq \int \left(\mathrm{E}\left|\sum_{i=1}^n \varepsilon_i \|f_i(t)\|\right|^p\right) d\mu(t)\\
&= \mathrm{E}\left(\int \left|\sum_{i=1}^n \varepsilon_i \|f_i(t)\|\right|^p d\mu(t)\right)\\
&= \mathrm{E}\left\|\sum_{i=1}^n \varepsilon_i F_i\right\|_{L^p(\mathbb{R})}^p\\
&\geqq \mathrm{E}\left|\sum_{i=1}^n \varepsilon_i \|F_i\|_{L^p(\mathbb{R})}\right|^p,
\end{aligned}$$

where the two inequalities above follow from the fact that $X$ and $L^p(\mathbb{R})$ are of Hanner cotype $p$ (the assumption on $X$ and Theorem 1) and $F_i$ is the real function $F_i(t) = \|f_i(t)\|$. This completes the proof.

PROPOSITION 1. Let $1 \leqq p \leqq r \leqq 2$. Then $L^r$ is of Hanner cotype $p$ and $L^p(L^r)$ is of Hanner cotype $p$.

PROOF. $L^r$ is isometrically imbeddable into $L^p$ since $1 \leqq p \leqq r \leqq 2$, so the assertion follows.

## References


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